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Mehmet Namik Oğuztöreli, Chiu Fan Lee

On a Diffusion Problem with a Time-Lag Concentration-Dependent Diffusion Coefficient

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Analisi matematica. — *On a Diffusion Problem with a Time-Lag Concentration-Dependent Diffusion Coefficient* (*). Nota di MEHMET NAMIK OĞUZTÖRELI e CHIU FAN LEE, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota viene studiato un problema di diffusione in un mezzo semi infinito con coefficiente di diffusione ritardato e funzione della concentrazione.

1. FORMULATION OF THE PROBLEM

In this paper we deal with the solution of a one-dimensional diffusion problem in a semi-infinite medium when the diffusion coefficient depends on the concentration by the equation

$$\frac{\partial \tilde{C}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\tilde{D}(\tilde{C}(x,t-h))\frac{\partial \tilde{C}(x,t)}{\partial x}\right] \tag{1.1}$$

for $x, t > 0$, subject to the conditions

$$\begin{cases} \tilde{C}(x,t) = a & \text{for } x > 0 \;, \quad -h \leq t \leq 0 \,, \\ \tilde{C}(x,t) = b & \text{for } x = 0 \;, \quad t > 0 \,, \end{cases} \tag{1.2}$$

where x and t are the spatial and time variables, h is a positive constant, a and b are certain given constants such that $b > a \geq 0$, $\tilde{C}(x,t)$ denotes the concentration and $\tilde{D}(\tilde{C})$ is the diffusion coefficient. We assume that the function $\tilde{D}(\tilde{C})$ is continuously differentiable in $\tilde{C}$ in some $\tilde{C}$–interval.

Although Problem (1.1)–(1.2) has attracted a great deal of attention of a number of scientists in the case $h = 0$, the authors did not encounter any contribution in the case $h > 0$. In the case $h = 0$ only a few formal solutions of certain very special cases have been obtained under the well known assumptions of Boltzmann. Relevant references can be found in [1]–[4]. Let us note that there are great difficulties in the finding of the solution of (1.1)–(1.2) in case $h = 0$, because of the non-linearity on the right-hand side of Equation (1.1) and of the discontinuity in the boundary condition (1.2). However, in the case $h > 0$, Problem (1.1)–(1.2) can be solved without any great difficulty. A method of solution of the time-lag problem will be presented in the next section.

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2. Solution by Successive Continuations

First we simplify Equations (1.1)–(1.2) by the transformation

$$C(x, t) = \frac{\tilde{C}(x, t) - a}{b - a} \tag{2.1}$$

which yields

$$\frac{\partial C(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[D(C(x, t-h))\frac{\partial C(x, t)}{\partial x}\right] \tag{2.2}$$

for $x, t > 0$, and

$$\begin{cases} C(x, t) = 0 & \text{for } x > 0 \ , \quad -h \leq t \leq 0 \ , \\ C(x, t) = 1 & \text{for } x = 0 \ , \quad t > 0 \ . \end{cases} \tag{2.3}$$

where

$$D(C) = \tilde{D}((b-a)\tilde{C} + a) \ . \tag{2.4}$$

Clearly, Problem (1.1)–(1.2) is equivalent to Problem (2.2)–(2.3). The solutions of these two problems can be constructed simultaneously by successive continuations from a time-interval $(n-1)h \leq t \leq nh$ to the interval $nh \leq t \leq (n+1)h$, $n = 0, 1, 2, \cdots$. In this way the original linear problem will be reduced to a piecewise linear problem. To do so we proceed as follows:

Since $C(x, t) = 0$ for $x > 0$, $-h \leq t \leq 0$ by Equation (2.3), we have

$$\frac{\partial C(x, t)}{\partial t} = D_0 \frac{\partial^2 C(x, t)}{\partial x^2} \tag{2.5}$$

for $x > 0$, $0 < t \leq h$, subject to the conditions

$$\begin{cases} C(x, 0) = 0 & \text{for } x > 0 \ , \\ C(0, t) = 1 & \text{for } 0 < t \leq h \ , \end{cases} \tag{2.6}$$

where

$$D_0 = D(0) \ . \tag{2.7}$$

It is well known (cfr. [1]) that the function

$$C_1(x, t) = 1 - (\pi D_0 t)^{-1/2} \int_0^{x/2\sqrt{D_0 t}} \exp\left\{-\frac{\xi^2}{4 D_0 t}\right\} d\xi \tag{2.8}$$

is the only solution of Equation (2.2) for $x > 0$, $0 \leq t \leq h$ satisfying the conditions (2.3). Hence we have

$$\frac{\partial C(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[D_1(x, t)\frac{\partial C(x, t)}{\partial x}\right] \tag{2.9}$$

for $x > 0, h < t \leq 2h$, subject to the condition

$$(2.10)\qquad \begin{cases} C(x, h) = C_1(x, h) & \text{for } x > 0 \\ C(0, t) = 1 & \text{for } h < t \leq 2h, \end{cases}$$

where

$$(2.11)\qquad D_1(x, t) = D(C_1(x, t - h)) \quad \text{for } x \geq 0 \quad , \quad h \leq t \leq 2h.$$

Since the function $D_1(x, t)$ is continuously differentiable with respect to x and t for $x > 0$ and $h < t \leq 2h$, the linear parabolic differential equation (2.9) has a unique solution for $x > 0$ and $h < t \leq 2h$ satisfying the conditions (2.10), which can be constructed by the method of parametrix (cfr. [5]). We denote this solution by $C_2(x, t)$. Thus we have

$$(2.12)\qquad \frac{\partial C(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[D_2(x, t)\frac{\partial C(x, t)}{\partial x}\right]$$

for $x > 0, 2h < t \leq 3h$, subject to the conditions

$$(2.13)\qquad \begin{cases} C(x, 2h) = C_2(x, 2h) & \text{for } x > 0, \\ C(0, t) \quad = 1 & \text{for } 2h < t \leq 3h. \end{cases}$$

where

$$(2.14)\qquad D_2(x, t) = D[C_2(x, t - h)] \quad \text{for } x \geq 0 \quad , \quad 2h \leq t \leq 3h.$$

Since $D_2(x, t)$ is continuously differentiable with respect to x and t for $x > 0, 2h < t \leq 3h$, Equation (2.12) has a unique solution $C_3(x, t)$ for $x \geq 0, 2h \leq t \leq 3h$ satisfying the conditions (2.13).

Now, let $C_n(x, t)$ be the solution of Problem (2.2)–(2.3) for $x \geq 0$, $(n - 1)h \leq t \leq nh$. Then, putting

$$(2.15)\qquad D_n(x, t) = D[C_n(x, t - h)] \quad \text{for } x \geq 0 \quad , \quad nh \leq t \leq (n + 1)h,$$

we find

$$(2.16)\qquad \frac{\partial C_{n+1}(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[D_n(x, t)\frac{\partial C_{n+1}(x, t)}{\partial x}\right]$$

for $x > 0, nh < t \leq (n + 1)h$, subject to the conditions

$$(2.17)\qquad \begin{cases} C_{n+1}(x, nh) = C_n(x, nh) & \text{for } x > 0 \\ C_{n+1}(0, t) \quad = 1 & \text{for } nh < t \leq (n + 1)h. \end{cases}$$

Clearly the linear parabolic differential equation (2.16) admits a unique solution $C_{n+1}(x, t)$ satisfying the conditions (2.17) which can be constructed by the method of parametrix. Thus the above described method of solution can be applied for any n.

In a subsequent paper we shall investigate the dependence of the solutions of Problem (1.1)–(1.2) on the time-lag h, and the approximation of the solutions of the non time-lag problem by the solutions of the time-lag problem.

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