Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Finite union of cyclic subgroups

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.1, p. 4–5.

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<http://www.bdim.eu/item?id=RLINA_1971_8_50_1_4_0>

Algebra. — *Finite union of cyclic subgroups.* Nota di Antonio Machì, presentata (*) dal Socio B. Segre.

Riassunto. — Si dimostra che un gruppo G che sia unione insiemistica di un numero finito di sottogruppi ciclici infiniti è necessariamente ciclico.

In this paper we prove the following Theorem:

Theorem. – *If a group* G *is the set-theoretic union of a finite number of infinite cyclic subgroups*:

$$G = H_1 \cup H_2 \cup \cdots \cup H_n$$

then G *is cyclic* (*so that* $n = 1$).

To prove this Theorem we make use of the following Lemma, which is already known; we repeat the proof here for the sake of completeness.

Lemma. – *Let* G *be a group and let* $H_1, H_2, \cdots, H_n$ *be a finite number of subgroups. Suppose there exists a finite collection of elements* $x_{ij} \in G$ ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, f(i)$) *with*

$$G = \bigcup_{i,j} H_i x_{ij}$$

(*set-theoretic union*). *Then for some* i, $[G : H_i] < \infty$.

Proof. By relabeling we can assume all the H_i to be distinct. Induct on n. The case $n = 1$ is clear. If a full set of cosets of H_n appears among the $H_n x_{nj}$ then H_n has finite index and we are finished. Otherwise if $H_n x$ is missing then $H_n x \subseteq \bigcup_{i,j} H_i x_{ij}$. But $H_n x \cap H_n x_{nj}$ is empty so $H_n x \subseteq \bigcup_{i \neq n, j} H_i x_{ij}$. Thus

$$H_n x_{nr} \subseteq \bigcup_{i \neq n, j} H_i x_{ij} x^{-1} x_{nr}$$

and G can be written as a finite union of cosets of $H_1, H_2, \cdots, H_{n-1}$. By induction, $[G : H_i] < \infty$ for some $i = 1, 2, \cdots, n - 1$ and the result follows.

The above Lemma has an immediate corollary:

Corollary. – *If a group* G *is the set-theoretic union of a finite number of subgroups, then one of them has finite index in* G.

Proof of the theorem. – By the Corollary to the Lemma, one of the H_i's has finite index in G. Assume H_1 is such a subgroup. By Poincaré's theorem [1], G has a normal subgroup K of finite index and $K \subseteq H_1$ (in fact, K is the intersection of all the conjugates of H_1); thus, G/K is a finite group.

(*) Nella seduta del 9 gennaio 1971.

We shall show now that every nonidentity subgroup of G has finite index in G; the result will then follow from a theorem of Fedorov [2]. Suppose $1 \neq H \leq G$; HK/K is finite, being a subgroup of G/K and this implies that $H/H \cap K$ is finite. But H is infinite, since G is torsion-free, and therefore $H \cap K \neq 1$. Since K is cyclic (being $K \subseteq H_1$, a cyclic group), the index of $H \cap K$ in K is finite and, $[G : K]$ also being finite, we have

$$[G : H \cap K] = [G : K][K : H \cap K] < \infty,$$

i.e., $H \cap K$ has finite index in G. Since $H \geq H \cap K$, the index of H in G is also finite. This completes the proof.

REFERENCES

[1] W. R. SCOTT, *Group Theory*, Prentice Hall Inc. (1964) Theorem 1.7.10.
[2] *Ibid.* Theorem 15.1.20.