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Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## On compact metrisable spaces

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RENDICONTI

DELLE SEDUTE

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**Classe di Scienze fisiche, matematiche e naturali**

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*Presiede il Presidente* BENIAMINO SEGRE

## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Matematica.** — *On compact metrizable spaces.* Nota di OFELIA TERESA ALAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO.. — Si assegnano condizioni atte ad assicurare la metrizzabilità di uno spazio di Hausdorff.

Let E be a topological space and U be a set of open coverings of E. We shall say that U is cofinal if for any open covering of E there is a covering belonging to U which refines it. We shall say that U is totally pre-ordered if for any two elements $\alpha$ and $\beta$ of U we must have that either $\alpha$ refines $\beta$ or $\beta$ refines $\alpha$.

In the present Note we shall solve a question raised by Professor R. G. Lintz, namely: Must a Hausdorff compact space with a set of open coverings, cofinal and totally pre-ordered, be metrizable?

THEOREM I.—*Let* E *be a Hausdorff paracompact space with a set of open coverings of* E *cofinal and totally pre-ordered. If there exists a non-open* $G_\delta$–*subset of* E, *then* E *is metrizable.*

*Proof.*—There is a point $b \in E$ and a sequence of open neighborhoods of $b$, $(V_n)_{n \geq 1}$, such that $\bigcap_{n=1}^{\infty} V_n$ is not a neighborhood of $b$.

Let U be a set of open coverings of E cofinal and totally pre-ordered. For any natural number $n \geq 1$, we fix an element $\alpha_n$ of U which refines the open covering $\{V_n, E - \{b\}\}$. The set $\{\alpha_n | n \geq 1\}$ is cofinal. Indeed,

(*) Nella seduta del 14 novembre 1970.

for any $\alpha \in U$ there is a natural number $n \geq 1$ such that $\alpha_n$ refines $\alpha$. Otherwise, since U is totally pre-ordered, $\alpha$ refines $\alpha_n$ for any $n \geq 1$; thus if $b \in X \in \alpha$, then $X \subset V_n$ for any $n \geq 1$, which is not possible.

Let $\gamma_1$ be a locally finite open covering of E which refines $\alpha_1$ and for any $n \geq 2$ (by induction) we fix a locally finite open covering of E, $\gamma_n$ which refines $\gamma_1, \cdots, \gamma_{n-1}$ and $\alpha_n$. The set $\{\gamma_n \mid n \geq 1\}$ is cofinal and totally pre-ordered. Furthermore, the set $\bigcup_{n=1}^{\infty} \gamma_n$ is an open basis for the topology on E. It follows from the Nagata-Smirnov Metrization Theorem ([1], p. 169) that E is metrizable.

COROLLARY 1.—*Let be a Hausdorff space with a set of open coverings of* E *cofinal and totally pre-ordered. If the set of all non-isolated points of* E *is either an infinite compact set or a finite* $G_\delta$*–set, then* E *is metrizable.*

*Proof.*—Let F be the set of all non-isolated points of E. If $F = \varnothing$, then E is a discrete space and, thus, metrizable. Suppose that $F \neq \varnothing$. E is a paracompact space because, in both cases, F is a compact subset of E. We shall prove that there exists a non-open $G_\delta$–subset of E.

If F is a finite nonempty $G_\delta$–subset of E, then we can write $F = \bigcap_{n=1}^{\infty} V_n$, where $V_n$ is open for any $n \geq 1$; moreover, for any $b \in F$, F is not a neighborhood of $b$.

Let K be an infinite compact subset of E and let $A = \{x_1, \cdots, x_n, \cdots\}$ be an infinite countable subset of K. If $b$ is an accumulation point of A, for any natural number $n \geq 1$, there is an open neighborhood of $b$, $V_n$, such that $x_n$ belongs to $V_n$ if and only if $x_n = b$. It follows that the set $\bigcap_{n=1}^{\infty} V_n$ is not a neighborhood of $b$.

Now, the thesis follows immediately from theorem 1. Cor. 1 implies:

COROLLARY 2.—*Let* E *be a Hausdorff space with a set of open coverings of* E *cofinal and totally pre-ordered. If* E *is compact, then* E *is metrizable.*

THEOREM 2.—*A metrizable space has a set of open coverings cofinal and totally pre-ordered if and only if the set of all non-isolated points is compact.*

*Proof.*—Let E be a metrizable space and let F denote the set of all non-isolated points of E. If $F = \varnothing$, then E is discrete and $U = \{\alpha\}$, where $\alpha = \{\{x\} \mid x \in E\}$, is a set of open coverings of E cofinal and totally pre-ordered. Suppose that $F \neq \varnothing$. Let $d$ be a distance for E.

First we shall prove that the condition is necessary. Let $U = \{\alpha_n \mid n \geq 1\}$ be an infinite countable set of locally finite open coverings of E cofinal and totally pre-ordered (it exists by virtue of the proof of theorem 1). If F is not compact, there is an infinite countable subset of F, A, with no accumulation point. Putting $A = \{x_1, \cdots, x_n, \cdots\}$, for any natural number $n \geq 1$, let $V_n$ be an open neighborhood of $x_n$, such that:

$$V_n \cap A = \{x_n\} \tag{1}$$

and

$$(2) \qquad V_n \subsetneq \cap \{X \in \alpha_n \mid x_n \in X\}.$$

There is no element of U which refines the open covering $\{V_n \mid n \geq 1\} \cup \cup \{E - A\}$. Thus F is compact.

Now we prove that the condition is sufficient. If F is compact, for any natural number $n \geq 1$, there is a finite subset of F, $F_n$, such that the union of the open balls $B(x, 1/n)$ when $x$ changes in $F_n$ contains F. Let us denote this union by $T_n$. The set $\alpha_n = \{B(x, 1/n) \mid x \in F_n\} \cup \{\{y\} \mid y \in E - T_n\}$ is a locally finite open covering of E for any $n \geq 1$. The set $\{\alpha_n \mid n \geq 1\}$ is cofinal and, by using the same argument applied in theorem 1, it follows that there is a set of open coverings of E cofinal and totally pre-ordered.

## REFERENCE

[1] W. J. PERVIN, *Foundations of General Topology*, «Academic Press», New York 1964.