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Matematica. — *Maximal monotonicity and m-accretivity of* A + B. Nota di BRUCE CALVERT (*), presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si danno condizioni su due operatori A e B entrambi massimali monotoni (rispettivamente m-accretivi) affinchè A + B sia massimale monotono (m-accretivo). L'ipotesi usuale che A sia limitato rispetto a B è sostituita dalla condizione più debole che A e B « puntino nella stessa direzione ». Quando uno degli operatori è il subgradiente di una funzione convessa si ottengono risultati più generali.

Let X be a Banach space over the reals R with dual X^*. The value of $x^* \in X^*$ at $x \in X$ will be denoted by either (x^*, x) or (x, x^*). A subset A of $X \times X^*$ is called monotone if for $[x, x^*]$ and $[u, u^*]$ in A we have

$$(x^* - u^*, x - u) \geq 0 .$$

A monotone set is maximal if it is not properly contained in another monotone set. Equivalently we regard A as a function from X to $P(X^*)$, subsets of X^*. Let A be a subset of $X \times X^*$.

One defines $Ax = \{z^* : [x, z^*] \in A\}$, $A^{-1}z^* = \{x : z^* \in Ax\}$, $D(A) = \{x : Ax \neq \emptyset\}$, $R(A) = \cup \{Ax : x \text{ in } X\}$, for α in R, $(\alpha A)x = \{\alpha y^* : y^* \text{ in } Ax\}$, $(A + B)x = \cup \{y^* + z^* : y^* \text{ in } Ax, z^* \text{ in } Bx\}$ for $B : X \to P(X^*)$. If C is a nonempty subset of X or X^*, one defines $|C| = \inf \{\|x\| : x \in C\}$.

If A is a subset of $X \times X$, or equivalently a function from X to $P(X)$, one defines Ax, A^{-1}, $D(A)$, $R(A)$, αA, $A + B$ similarly. Then A is accretive if for all $\lambda > 0$ $(I + \lambda A)^{-1}$ is nonexpansive, i.e. for $[x, y]$ and $[u, v]$ in A,

$$\|(x + \lambda y) - (u + \lambda v)\| \geq \|x - u\| .$$

A is m-accretive if also $R(I + \lambda A) = X$ for $\lambda > 0$. Conditions of relative boundedness have been given for the sum A + B of two nonlinear maximal monotone [3, Th 2.3] or m-accretive [7, Th 9.22], [11, Th 10.2], [12, Th 4.2] operators to have the same property. The idea of this paper is that Ax and Bx should point in the same direction for x in $D(A) \cap D(B)$. In other words, just as monotonicity and accretivity are directional rather than boundedness properties, perturbation theorems for monotone and accretive operators may be given under directional hypotheses. We suppose $f : X \to (-\infty, \infty]$ is convex, not identically ∞, and lower semicontinuous. Then $\partial f : X \to P(X^*)$, the subdifferential of f, is defined by $w^* \in \partial f(x)$ iff for all y in X

$$f(y) \geq (w^*, y - x) + f(x) .$$

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Then, [14], ∂f is maximal monotone. Browder [8] asks for conditions on maximal monotone A for $A + \partial f$ to be maximal monotone. These are given in Theorem 2.

The subdifferential of $f(x) = \|x\|^2/2$ is denoted by J, and called the duality map. Similar results to this paper would arise if we took f to be other functions of the norm as given in e.g. [6]. We recall the following theorem of Browder [5, 6].

Let X be a reflexive Banach space with X, X^* strictly convex. Let $A : X \to P(X^*)$ be monotone. Then A is maximal monotone iff $R(J+A)=X^*$. We recall the following theorem of Brezis-Crandall-Pazy [3].

Let X be a reflexive Banach space with X, X^* strictly convex. Let $A : X \to P(X^*)$ and $B : X \to P(X^*)$ be maximal monotone. By Browder's theorem, given $\lambda > 0$, z in X, there exists a unique $[z_\lambda, z_\lambda^*] \in A$ with $J(z_\lambda - z) + \lambda z_\lambda^* = 0$. Defining $A_\lambda : X \to X^*$ by $z_\lambda^* = A_\lambda(z)$, by [3] and [5] $B + A_\lambda$ is maximal monotone, so that by Browder's theorem, given f^* in X^* there exists a unique x in X such that

$$Jx_\lambda + A_\lambda x_\lambda + Bx_\lambda \ni f^*. \tag{1}$$

Then $f^* \in R(J + A + B)$ iff $\|A_\lambda x_\lambda\|$ is bounded as $\lambda \to 0$.

THEOREM 1: *Let* X *be a reflexive Banach space with* X, X^* *strictly convex. Suppose* A *and* B *from* X *to* $P(X^*)$ *are both maximal monotone. Suppose*

$$(I + \lambda J^{-1} A)^{-1} D(B) \subset D(B) \qquad \text{for} \quad \lambda > 0. \tag{2}$$

Suppose $k(r)$, $c(r)$ *and* $d(r)$ *are continuous functions of* r, $k(r) < 1$ *for every* r, $r^2/d(r) \to \infty$ *when* $r \to \infty$ *such that for* x *in* $D(B) \cap D(A)$ *and* x^* *in* Ax *there exists* y^* *in* Bx *such that*

$$(y^*, J^{-1} x^*) \geq - k(\|x\|) \|x^*\|^2 - c(\|x\|)\, d(\|x^*\|). \tag{3}$$

Then $A + B : X \to P(X^*)$ *is maximal monotone.*

Proof: It follows from (2) that there exists $\tilde{x}$ in $D(A) \cap D(B)$, and letting $\tilde{A}(x) = A(x + \tilde{x})$, $\tilde{B}(x) = B(x + \tilde{x})$ we have $0 \in D(\tilde{A}) \cap D(\tilde{B})$. Furthermore (2) and (3) hold for $\tilde{A}$ and $\tilde{B}$, after changing k and c. Hence, we may assume $0 \in D(A) \cap D(B)$. By Browder's theorem we have to show $R(A + B + J) = X$. Consequently it suffices to show that given f^*, the $A_\lambda x_\lambda$ in (1) are bounded as $\lambda \to 0$. We set $v_\lambda = (I + \lambda J^{-1} A)^{-1} x_\lambda$. Then v_λ is in D(B) by (2). Also $A_\lambda x_\lambda$ is in Av_λ. Take d_λ^* in Bv_λ such that (3) gives

$$(d_\lambda^*, J^{-1} A_\lambda x_\lambda) \geq - k(\|v_\lambda\|) \|A_\lambda x_\lambda\|^2 - c(\|v_\lambda\|)\, d(\|A_\lambda x_\lambda\|).$$

Suppose b_λ^* is the element of $B(x_\lambda)$ giving equality in (1), that is

$$A_\lambda x_\lambda + b_\lambda^* + Jx_\lambda = f^*. \tag{4}$$

Since B is monotone, $(d^*_\lambda - b^*_\lambda, v_\lambda - x_\lambda) \geq 0$. We take the product of (4) with $J^{-1} A_\lambda x_\lambda = \lambda^{-1}(x_\lambda - v_\lambda)$.

$$\|A_\lambda x_\lambda\|^2 = (f^* - Jx_\lambda - b^*_\lambda, J^{-1} A_\lambda x_\lambda)$$
$$= (d^*_\lambda - b^*_\lambda, \lambda^{-1}(x_\lambda - v_\lambda)) + (f^* - Jx_\lambda - d^*_\lambda, J^{-1} A_\lambda x_\lambda),$$

$$\|A_\lambda x_\lambda\|^2 \leq \|f^* - Jx_\lambda\| \, \|A_\lambda x_\lambda\| + k(\|v_\lambda\|) \, \|A_\lambda x_\lambda\|^2 + c(\|v_\lambda\|) \, d(\|A_\lambda v_\lambda\|). \tag{5}$$

We claim x_λ is bounded as $\lambda \to 0$, taking the product of (4) with x_λ, and taking b^* in B (0)

$$\|x_\lambda\|^2 \leq (x_\lambda - 0, J(x_\lambda) - J(0)) + (x_\lambda - 0, A_\lambda x_\lambda + b^*_\lambda - A_\lambda 0 - b^*)$$
$$\leq \|x_\lambda\| (\|f^*\| + \|A_\lambda(0)\| + \|b^*\|).$$

Since $\|A_\lambda(0)\| \leq |A(0)|$ by Lemma 1.3 *(d)* of [3], after dividing by $\|x_\lambda\|$ we have $\|x_\lambda\| \leq M$. We claim v_λ is bounded as $\lambda \to 0$. Taking the product of $\lambda A_\lambda x_\lambda = J(x_\lambda - v_\lambda)$ with v_λ gives $(J(x_\lambda - v_\lambda), v_\lambda) \geq \lambda(a^*, v_\lambda)$ for any a^* in A (0), hence $\|x_\lambda - v_\lambda\|^2 \leq \lambda(a^*, x_\lambda - v_\lambda) + (J(x_\lambda - v_\lambda), x_\lambda) - \lambda(a^*, x_\lambda)$ and consequently $\|x_\lambda - v_\lambda\|^2 - (|A(0)| + M)\|x_\lambda - v_\lambda\| - |A(0)| M \leq 0$, for $\lambda \leq 1$, which implies $\|x_\lambda - v_\lambda\|$ is bounded, and consequently v_λ is bounded. From (5) it now follows that $A_\lambda x_\lambda$ is bounded as $\lambda \to 0$. q.e.d.

Remark 1: Two special cases of (3) are:

$$(y^*, J^{-1} x^*) \geq 0, \tag{3'}$$

$$\|y^*\| \leq k(\|x\|) \|x^*\| + c(\|x\|), \tag{3''}$$

$$\text{i.e. } |Bx| \leq k(\|x\|) |Ax| + c(\|x\|),$$

which is Theorem 2.3 of [3]. We note in [3, Theorem 2.3] the approximations B_λ are taken on B rather than A. In Theorem 3.2 of [3], to show that $-\Delta + \partial\psi_K$ is maximal monotone, a calculation like that in Theorem 1 is used.

Also, in Theorem 3.1 of [3] it is the condition (3′) that gives $-\Delta + \bar{\beta}$ maximal monotone. In [4] it is supposed that ∂f and B are maximal monotone and satisfy a condition like (3″) in Hilbert space, and shown that the semigroup satisfies regularity conditions.

COROLLARY 1: *Suppose* X *a reflexive Banach space,* $A, B : X \to P(X^*)$ *both maximal monotone,* B^{-1} *or* A *being locally bounded,* $R(A) \subset D(B)$. *If* BA *is accretive then it is m–accretive.*

Proof: Let J be the duality map for an equivalent norm making X, X^* strictly convex [1]. By the proof of Theorem 2 of [10] it suffices to show $A^{-1} + B$ is maximal monotone. We define $\tilde{A}, \tilde{B}$ by $\tilde{A}(x) = A(x + \tilde{x})$ and $\tilde{B}(x^*) = B(x^*) - \tilde{w}$ where $[\tilde{x}, \tilde{w}] \in BA$. $\tilde{B}\tilde{A}$ is accretive and $[0, 0] \in \tilde{B}\tilde{A}$. Consequently we may assume $[0, 0] \in BA$. Hence for b in $B(x^*)$ and x^* in Ax, $(Jx, b) \geq 0$. This is condition (3′) of Remark 1. Condition (2) follows from $R(A) \subset D(B)$. q.e.d.

We now turn to the case where $A = \partial\psi_K$. Suppose K is a closed convex subset of a Banach space, the indicator function ψ_K is defined to be zero on K and ∞ elsewhere. We recall that if f is a lower semicontinuous function from X to $(-\infty, \infty]$, we say w is in $\partial f(x)$ if for all y in X

$$f(y) \geq (w, y - x) + f(x), \tag{6}$$

and ∂f is maximal monotone if f is not identically ∞. Consequently $\partial\psi_K$ is maximal monotone for K nonempty.

COROLLARY 2: *Let* X *be a Banach space with* X *and* X^* *strictly convex and* K *a closed convex subset of* X. *For* x *in* X *let* Px *be the nearest point to* x *on* K. *Suppose* $B: X \to P(X^*)$ *is maximal monotone,* $P(D(B)) \subset D(B)$, *and for* y *in* $D(B)$ *there exists* b^* *in* BPy *such that*

$$(b^*, y - Py) \geq -k(\|Py\|)\, \|y - Py\|^2 - c(\|Py\|)\, d(\|y - Py\|)$$

where k, c, d *are as in Theorem* 1. *Then* $B + \partial\psi_K$ *is maximal monotone.*

LEMMA 1: *Suppose* X *a reflexive Banach space with* X *and* X^* *strictly convex,* K *a closed convex nonempty subset of* X, P *the projection taking* x *to the nearest point on* K. *Let* ψ_K *be the indicator function of* K *and let* $\lambda > 0$ *be given. Then* $(I + \lambda J^{-1}\partial\psi_K)^{-1} = P$.

Proof: The Lemma 3.8 of [9] showed $(I + \lambda J^{-1}\partial\psi_K)^{-1} x \supset Px$ for X^* strictly convex, Px being the set of nearest points to x on K. When X is strictly convex the left hand side has only one element, giving equality. q.e.d.

Proof of Corollary 2: By the Lemma $P(D(B)) \subset D(B)$ gives (2) of Theorem 1. For (3), given x in $K \cap D(B)$ and x^* in $\partial\psi_K(x)$, let $y = x + J^{-1}x^*$, and take for y^* of (3) the b^* given in the statement of the corollary. q.e.d.

THEOREM 2: *Suppose* X *a reflexive Banach space with* X, X^* *strictly convex. Suppose* $A: X \to P(X^*)$ *is maximal monotone,* $f: X \to (-\infty, \infty]$ *is convex and lower semicontinuous,* k, c, d *are functions as given in Theorem 1. Suppose that for* $[v, a^*]$ *in* A, $\lambda > 0$

$$f(v + \lambda J^{-1} a^*) \geq f(v) - \lambda\,(k(\|v\|)\, \|a^*\|^2 + c(\|v\|)\, d(\|a^*\|)). \tag{7}$$

Then $\partial f + A$ *is maximal monotone.*

Proof: By (7) we may take $\tilde{x}$ in $D(A) \cap D(f)$, i.e. $f(\tilde{x}) < \infty$.

Let $\tilde{f}(v) = f(v + \tilde{x})$ and $\tilde{A}(v) = A(v + \tilde{x})$. Then $\partial f(x + \tilde{x}) = \partial\tilde{f}(x)$. Consequently we must show $\tilde{A} + \partial\tilde{f}$ is maximal monotone, or $R(\tilde{A} + \partial\tilde{f} + J) = X^*$, by Browder's Theorem. But $0 \in D(\tilde{A}) \cap D(\tilde{f})$, and (7) holds with A and f replaced by $\tilde{A}$ and $\tilde{f}$. Consequently we may assume $0 \in D(f) \cap D(A)$. We need to show that $A_\lambda x_\lambda$ given in (1) is bounded as $\lambda \to 0$. Suppose

$$A_\lambda x_\lambda + \partial f(x_\lambda) + J(x_\lambda) \ni f^*.$$

This means by (6) that for v in X,

$$f(v) \geq (f^* - A_\lambda x_\lambda - Jx_\lambda, v - x_\lambda) + f(x_\lambda). \tag{8}$$

Letting $v_\lambda = (I + \lambda J^{-1}A)^{-1} x_\lambda$, in particular

$$f(v_\lambda) \geq (f^* - A_\lambda x - Jx_\lambda, -\lambda J^{-1} A_\lambda x_\lambda) + f(x_\lambda).$$

But by (7),

$$f(v_\lambda) \leq f(x_\lambda) + \lambda (k(\|v_\lambda\|) \|A_\lambda x_\lambda\|^2 + c(\|v_\lambda\|) d(\|A_\lambda x_\lambda\|)).$$

After dividing by λ, we have

$$\|A_\lambda x_\lambda\|^2 \leq k(\|v_\lambda\|) \|A_\lambda x_\lambda\|^2 + c(\|v_\lambda\|) d(\|A_\lambda x_\lambda\|) + \|f^* - Jx_\lambda\| \|A_\lambda x_\lambda\|,$$

which yields (5).

We now let $v = 0$ in (8), and obtain

$$\|x_\lambda\|^2 \leq -(A_\lambda x_\lambda, x_\lambda) + \|x_\lambda\| \|f^*\| + f(0) - f(x_\lambda)$$

Take $[y, w^*] \in \partial f$. Since $(A_\lambda x_\lambda, x_\lambda) \geq (A_\lambda 0, x_\lambda)$ and $f(x_\lambda) \geq (w^*, x_\lambda - y) + f(y)$, it follows that

$$\|x_\lambda\|^2 \leq \|x_\lambda\| (\|f^*\| + \|w^*\| + \|A_\lambda(0)\|) + f(0) - f(y) - (w^*, y).$$

Since $A_\lambda(0)$ are bounded, it follows that x_λ are bounded.

As in Theorem 1, v_λ are bounded as $\lambda \to 0$, and hence $A_\lambda x_\lambda$ are bounded. q.e.d.

Remark 2: A special case of (7) is that for x in X and $\lambda > 0$

(7′) $$f((I + \lambda J^{-1} A)^{-1} x) \leq f(x).$$

One sees that Theorems 1 and 2 are similar, and if (7) implied (2) and (3) (or more simply (7′) implied (2) and (3′)) then Theorem 1 would imply Theorem 2. However, this is not true in general, although the converse holds.

Remark 3: Suppose X, A and f are as in Theorem 2, without (7). Suppose for $\lambda > 0$, $(I + \lambda J^{-1}A)^{-1} D(\partial f) \subset D(\partial f)$, and for v in $D(A) \cap D(\partial f)$ and a^* in Av there exists b^* in $\partial f(v)$ such that $(b^*, J^{-1}(a^*)) \geq 0$. Then (7′) holds.

Proof: Suppose a^* in Av, $\lambda > 0$, and $v + \lambda J^{-1} a^* = x$. We want to show $f(v) \leq f(x)$. If $x \in D(\partial f)$ so does v by assumption, and by (6), for all b^* in $\partial f(v)$, $f(v + \lambda J^{-1} a^*) \geq f(v) + (b^*, \lambda J^{-1} a^*)$. By assumption there exists b^* in $\partial f(v)$ making $(b^*, J^{-1}a^*) \geq 0$, giving $f(v) \leq f(x)$.

Suppose now $f(x) < \infty$, we claim there exists $x_n \to x$, $f(x_n) \to f(x)$, $x_n \in D(\partial f)$. This is because if K is the epigraph of f, $K = \{(x, k) \in X \times R : k \geq f(x)\}$ it has supporting hyperplanes in $X \times R$ at a dense subset of the boundary by the Bishop Phelps theorem [2]. Now $(x, f(x))$ is in the boundary, and the supporting hyperplanes give x_n in $D(\partial f)$ with $x_n \to x$, $f(x_n) \to f(x)$ [14]. Given $\lambda > 0$, $(I + \lambda J^{-1} A)^{-1} x_n$ converges weakly to $(I + \lambda J^{-1} A)^{-1} x$ by Lemma 1.3 (*c*) of [3]. Since $x_n \in D(\partial f)$, $f((1 + \lambda J^{-1}A)^{-1} x_n) \leq f(x_n)$. Since f is lower semicontinuous, $f(v) = f((1 + \lambda J^{-1}A)^{-1} x) \leq \varliminf f((1 + \lambda J^{-1}A)^{-1} x_n) \leq \varliminf f(x_n) = f(x)$. If $f(x) = \infty$, then $f(v) \leq f(x)$. q.e.d.

COROLLARY 1: *Suppose* X *a reflexive Banach space*, X, X^* *strictly convex*, K *a closed convex nonempty subset of* X. *For* x *in* X *let* Px *be the nearest point to* x *on* K. *Suppose, with* c *a function as in Theorem* 1, *for* x *in* X

$$f(x) \geq f(\mathrm{P}x) - c(\|\mathrm{P}x\|)(|\mathrm{K}-x|^2 + |\mathrm{K}-x|)$$

Then $\partial f + \partial\psi_{\mathrm{K}}$ *is maximal monotone, and equal to* $\partial(f+\psi_{\mathrm{K}})$.

Proof: By Lemma 1, with $\mathrm{A} = \partial\psi_{\mathrm{K}}$, the condition (7) fails but the proof of Theorem 2 gives $\partial f + \partial\psi_{\mathrm{K}}$ maximal monotone. Since $\partial f + \partial\psi_{\mathrm{K}} \subset \partial(f+\psi_{\mathrm{K}})$ and $\partial(f+\psi_{\mathrm{K}})$ is monotone, the maximality gives equality. q.e.d.

COROLLARY 2: *Suppose* X *a reflexive Banach space with* X, X^* *strictly convex*, K *and* P *as in Corollary* 1, *and* B *another closed convex nonempty subset of* X. *If* $\mathrm{P}(\mathrm{B}) \subset \mathrm{B}$ *then* $\partial\psi_{\mathrm{K}} + \partial\psi_{\mathrm{B}} = \partial\psi_{\mathrm{B}\cap\mathrm{K}}$.

Proof: We take $f = \psi_{\mathrm{B}}$ in Corollary 1. $\mathrm{P}(\mathrm{B}) \subset \mathrm{B}$ implies $f(x) \geq f(\mathrm{P}x)$. By Corollary 1, $\partial\psi_{\mathrm{K}} + \partial\psi_{\mathrm{B}}$ is maximal monotone, and one sees $\psi_{\mathrm{B}} + \psi_{\mathrm{K}} = \psi_{\mathrm{B}\cap\mathrm{K}}$ q.e.d.

We will suppose X has uniformly convex dual X^*. Since $\mathrm{A}: \mathrm{X} \to P(\mathrm{X})$ is accretive iff y in Ax and v in Au implies $(y-v, \mathrm{J}(x-u)) \geq 0$ [7], [11], [12], the sum of two accretive operators is accretive.

THEOREM 3: *Suppose* X *a Banach space with* X^* *uniformly convex. Suppose* A *and* B *are m–accretive, and for* $\lambda > 0$

$$(\mathrm{I}+\lambda\mathrm{A})^{-1}\,\mathrm{D}(\mathrm{B}) \subset \mathrm{D}(\mathrm{B}). \tag{9}$$

Suppose for v *in* X *there is a neighborhood* N(v) *of* v, *a function* d *such that* $r^2/d(r) \to \infty$ *when* $r \to \infty$, *and* k *in* $[0, 1)$; *such that for* x *in* $\mathrm{D}(\mathrm{A}) \cap \mathrm{D}(\mathrm{B}) \cap \mathrm{N}(v)$ *and* a *in* Ax *there exists* b *in* Bx *such that*

$$(\mathrm{J}a, b) \geq -k\|a\|^2 - d(\|a\|). \tag{10}$$

Then A + B *is m–accretive.*

Proof. It is enough to alter the proof of [11, Th. 10.2] by taking approximations with A_λ instead of B_λ. One uses the same calculation as in Theorem 1 to show that if $\mathrm{A}_\lambda x_\lambda + \mathrm{B}x_\lambda + x_\lambda \ni y$ and $x_\lambda \to x_0$ as $\lambda \to 0$ then $\mathrm{A}_\lambda x_\lambda$ is bounded, the equivalent of [11, 10.8] where $\mathrm{B}_\lambda v_\lambda$ are shown to be bounded. q.e.d.

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