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Profile Problems for Transonic Flows with Shocks

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica applicata. — *Profile Problems for Transonic Flows with Shocks* (*). Nota di CATHLEEN S. MORAWETZ, presentata (**) dal Corrisp. G. FICHERA.

RIASSUNTO. — Si considera il problema consistente nel determinare un flusso attraverso un profilo simmetrico bidimensionale con un urto debole a termine del flusso supersonico. Viene anche formulato e studiato un problema di perturbazione corrispondente ad un cambiamento infinitesimale della velocità all'infinito. Viene provato un teorema di unicità per un analogo problema relativo all'equazione di Tricomi e, infine, formulato il problema di perturbazione relativo all'urto debole prodotto in un flusso privo di urti da un cambiamento della velocità.

§ 1. INTRODUCTION AND SUMMARY

In this paper we examine the problem of determining a flow past a two-dimensional symmetric profile with a weak shock terminating the supersonic flow, see [1]. A perturbation problem corresponding to an infinitesimal change in speed at infinity is also formulated and investigated. A uniqueness theorem for an analogous problem for the Tricomi equation is proved.

We are led finally to a formulation of the perturbation problem for a smooth transonic flow, i.e., for the weak shock induced in a shockless flow by a change of speed. This involves finding a solution of the perturbation equations along with boundary conditions and an appropriate singularity at the downstream intersection of the sonic line and the profile.

It is clear from earlier work [2, 3], that we cannot expect the existence of a solution without the admission of a sonic singularity. In fact it was established there that the perturbation problem is not well posed if the solution is sufficiently smooth. In [4] the analogous paradox has been resolved for a Dirichlet problem for the Tricomi equation.

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(**) Nella seduta del 14 novembre 1970.

§ 2. Formulation of the Shock Problem

The velocity (u, v) of a two-dimensional flow satisfies

$$(1) \qquad (\rho u)_x + (\rho v)_y = 0 \quad , \quad u_y - v_x = 0$$

where the density ρ is a known function of $q^2 = u^2 + v^2$, given the equation of state and Bernouilli's law. This relation has the property that $\mathrm{d}(\rho q)/\mathrm{d}q$ changes sign when the flow is sonic, $q = c_*$. The shock conditions, entropy and rotationality changes being neglected, are:

$$(2) \qquad [\rho u]\,\mathrm{d}y - [\rho v]\,\mathrm{d}x = 0 \quad , \quad [u]\,\mathrm{d}x + [v]\,\mathrm{d}y = 0$$

where $\mathrm{d}y/\mathrm{d}x$ is the slope of the shock and [ ] indicates the discontinuity. Thus we have

$$(3) \qquad [\rho u]\,[u] + [\rho v]\,[v] = 0 .$$

With $u = q \cos\theta$, $v = q \sin\theta$ and using $\sim$ to denote the average of the front and back state we have

$$(\widetilde{\rho q}\,[\cos\theta] + [\rho q]\,\widetilde{\cos\theta})\,(\tilde{q}\,[\cos\theta] + [q]\,\widetilde{\cos\theta})$$

$$+ (\widetilde{\rho q}\,[\sin\theta] + [\rho q]\,\widetilde{\sin\theta})\,([q]\,\widetilde{\sin\theta} + \tilde{q}\,[\sin\theta]) = 0$$

or

$$2\,\widetilde{\rho q}\,\tilde{q}\,(1 - \cos[\theta]) + \frac{1}{2}\,[\rho q]\,[q]\,(1 + \cos[\theta]) = 0$$

or to third order in $[\theta]$ or $[q]$

$$[\theta]^2 = -\frac{[\rho q]\,[q]}{\widetilde{\rho q}\,\tilde{q}} .$$

Noting that $\mathrm{d}(\rho q)/\mathrm{d}q = 0$ at $q = c_*$, $\rho = \rho_*$ we find to second order in $\delta q = q - c_*$

$$(4) \qquad [\theta] = \mathrm{A}_1\,\widetilde{\delta q}^{1/2}\,[\delta q]$$

where

$$\mathrm{A}_1^2 = -(\rho_*\,c_*^2)^{-1}\,\frac{\mathrm{d}^2}{\mathrm{d}q^2}(\rho q)\Big|_{q=c_*} .$$

We note for later use that we also find to second order

$$(5) \qquad \frac{[\rho u]}{[v]} = \frac{[\rho v]}{[u]} = \rho_*\,c_*\,\mathrm{A}_1\,\widetilde{\delta q}^{1/2} .$$

The simplified shock polar (4) must be coupled with (2) to determine the slope of the shock.

If we introduce the streamfunction ψ by $\psi_x = -\rho v$, $\psi_y = \rho u$ and the potential φ by $\varphi_x = u$, $\varphi_y = v$, we find that (2) reduces to

$$(6) \qquad \psi, \varphi \quad \text{continuous}$$

The conditions for a given speed q_∞ in the x–direction at ∞ and no circulation are

$$(7) \qquad \varphi_x \to q_\infty \quad , \quad \varphi_y \to 0 , \quad \varphi \ \text{single-valued.}$$

The condition on a fixed boundary is

$$\psi = \text{constant}. \tag{8}$$

The flow in a suitable hodograph plane with $\sigma = -\int \frac{\rho}{q}\, dq$, $K(\sigma) = -q\left(\frac{1}{\rho q}\right)_\sigma$ satisfies

$$K\psi_{\theta\theta} + \psi_{\varphi\varphi} = 0 \tag{9}$$

or

$$K\psi_\theta = \varphi_\sigma \quad , \quad \psi_\sigma = -\varphi_\theta . \tag{9*}$$

The conditions at ∞ reduce to describing the flow near $\sigma = \sigma_\infty$, $\theta = 0$. In the symmetric case to which we limit ourselves we have, see [11],

$$\psi - i\beta\varphi \sim \text{const.}\,(\theta + i(\sigma - \sigma_\infty)\beta)^{-1/2} \tag{10}$$

where $\beta^{-2} = K(\sigma_\infty)$. Corresponding to the stagnation points we have for ψ as $\sigma \to \infty$,

$$e^{2\sigma}\psi_\theta \quad , \quad e^{2\sigma}\psi_\sigma \quad \text{bounded} \tag{11}$$

where density has been normalized so that $q \sim e^\sigma$.

Across the shock, (6) becomes

$$\psi(\theta_1, \sigma_1) = \psi(\theta_2, \sigma_2) \quad , \quad \varphi(\theta_1, \sigma_1) = \varphi(\theta_2, \sigma_2) \tag{12}$$

where θ_1, θ_2, σ_1, σ_2 are related through (4) by, with $\gamma = -(\rho q)_{\sigma\sigma}/\rho_*^2\, c_*$,

$$(\theta_1 - \theta_2)^2 = -\frac{1}{2}\gamma(\sigma_1 + \sigma_2)(\sigma_1 - \sigma_2)^2. \tag{13}$$

The hodograph plane is illustrated in fig. 1. The curves C_1 and C_2 represent the two sides of the shock. The supersonic side C_2 must be " time-like " if the streamline boundary is " space-like ". And hence lies between the two

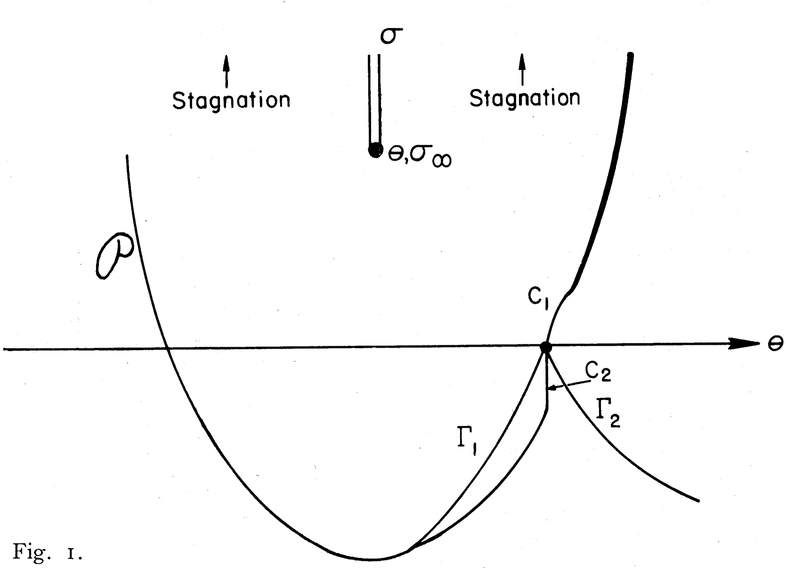

Fig. 1.

characteristics Γ_1 and Γ_2 as it crosses the sonic line. That is, if, say, the state ahead of the shock (2) tends to sonic, then $\sigma \to 0$ and

$$(\theta_1 - \theta_2)^2 = o(\sigma_1^3).$$

§ 3. Finding Profile Flows with Shocks

To find a profile flow in the hodograph plane we must find a solution of (8–13). For the construction of smooth flows see [5], and for local shock flows see [6]. There remains the problem of mapping such a flow into the physical plane. For this we would like to have the Jacobian $\frac{\partial(\psi, \varphi)}{\partial(\theta, \sigma)} = K\psi_\theta^2 + \psi_\sigma^2 \neq 0$. It is worth noting first that there does not exist a solution ψ with continuous second derivatives in θ, σ that does not have a limiting line which makes the mapping impossible. The argument is as follows: Expand ψ, φ in Taylor series about the sonic point of the shock, taking the flow there in the x–direction. Substitute the series in θ, σ in the shock conditions (12) and the differential equations (9) and (9*). Then up to second order with the derivatives evaluated at (0, 0),

$$\psi_\theta [\theta] + \psi_\sigma [\sigma] + \psi_{\theta\theta} \tilde{\theta} [\theta] + \psi_{\sigma\sigma} (\tilde{\theta} [\sigma] + \tilde{\sigma} [\theta]) = 0 ,$$

$$\psi_\sigma [\theta] - \psi_{\theta\sigma} \tilde{\theta} [\theta] + k\psi_\theta \tilde{\sigma} [\sigma] = 0 ,$$

$$k = K' = c^* \frac{d^2}{d\sigma^2} \left(\frac{1}{\rho q}\right)\bigg|_0 = -(\rho q)_{\sigma\sigma}/\rho_*^2 \, c_*^2 = \gamma .$$

Using (13) we see immediately that $\psi_\sigma = 0$ and also that

$$\pm (\gamma\tilde{\sigma})^{1/2} (\psi_\theta + \psi_{\theta\theta} \tilde{\theta} + \psi_{\theta\sigma} \tilde{\sigma}) + \psi_{\theta\sigma} \tilde{\tilde{\theta}} = 0 \quad , \quad \pm (\gamma\tilde{\sigma})^{1/2} \psi_{\theta\sigma} \tilde{\theta} + k\psi_\theta \tilde{\sigma} = 0 .$$

Thus since $\theta_1 \sim \theta_2$ we find $\theta_2 \sim \pm \psi_\theta (\gamma\tilde{\sigma})^{1/2}/\psi_{\theta\sigma}$.

Since one side of the shock is subsonic and the other supersonic $|\sigma| < |\sigma_2|$ by (13) and $\gamma > 0$ so that $|\theta_2| < |\psi_\theta/\psi_{\theta\sigma}| \; |\gamma\sigma_2|^{1/2}$.

But the limiting line is $\psi_\theta \varphi_\sigma - \psi_\sigma \varphi_\theta = K(\sigma) \psi_\theta^2 + \psi_\sigma^2 = 0$ or $\gamma\sigma\psi_\theta^2 + \psi_{\theta\sigma}^2 \theta^2 = 0$. Hence $|\theta_2| < |\theta|$ where θ_2 is on the shock curve and θ is on the limiting line. Thus the limiting line cuts off the shock at the subsonic region unless $\psi_\theta = 0$. In the latter case the higher expansion yields a flow with even worse mapping properties.

To avoid this difficulty we may either seek a solution with a singularity or settle for a solution that satisfies the shock conditions only to first order, see Garabedian and Korn [5].

We turn to the problem of finding a profile flow with an appropriate singularity in the hodograph plane, i.e., a solution of (8–13) with a given shape for the hodograph of the body (1) and a given shape for the subsonic

(1) We assume θ changes monotonically and σ has one minimum.

hodograph of the shock C_1. At this stage we can only check that the given data " count " correctly.

We examine the intersection of the characteristic Γ_1 (upstream) with $\mathcal{S}$ the profile hodograph. If this point is not downstream of the point of maximum speed, then using the methods described in [1] a flow solution of (9*) or (9) satisfying (8), (10), (11) with, say, $\psi = f$ on the subsonic shock hodograph C_1 is uniquely determined up to the characteristic Γ_1. The continuation of the solution for both φ and ψ up to the downstream characteristic Γ_2 is also uniquely determined, i.e., by one datum each on the characteristic Γ_1 and the space-like curve $\mathcal{S}$. This statement is almost certainly true without the restriction on Γ_1. In fact one may reasonably expect that the solution (φ, ψ) may be expressed as a linear functional of the singularity at $(0, \sigma_\infty)$ and the given function f. Thus whatever the timelike curve $C_2 : \sigma_2 = \sigma_2(\theta_2)$, representing the supersonic side of the shock both φ and ψ are determined on it.

Substituting these functional relations into the three shock conditions we find that we have two linear functional relations for the two functions f and $\sigma = \sigma_2(\theta_2)$. The singularity at $(0, \sigma_\infty)$ provides the non-homogeneity if we regard these two relations as equations for f and $\sigma_2(\theta_2)$, and we may expect that a solution can be determined. On the basis of the example described in [4] and the previous arguments here, there probably is a singularity at the sonic point of the shock.

§ 4. FORMULATION OF THE PERTURBATION PROBLEM FOR THE SPEED AT INFINITY

In this section we assume there exists a flow corresponding to some hodograph solution, such as that described in the preceding section. The equations for the perturbed potential $\delta\varphi$ and streamfunction $\delta\psi$ satisfy, see [11],

$$K(\sigma)\,\delta\varphi_\theta = \delta\psi_\sigma \quad , \quad \delta\varphi_\sigma = -\,\delta\psi_\theta \tag{14}$$

where now

$$K(\sigma) = \rho^2 (1 - q^2/c^2) \quad , \quad \sigma = -\int \frac{dq}{\rho q}\,. \tag{15}$$

Since the profile is undisturbed we have $\delta\psi = 0$ there, see (8), and hence

$$K\delta\varphi_\theta\, d\sigma - \delta\varphi_\sigma\, d\theta = 0 \quad \text{on} \quad \mathcal{S}. \tag{16}$$

The singularity at $(0, \sigma_\infty)$ corresponding to the perturbation of the speed δq_∞ is given by

$$\beta_1\,\delta\varphi_\sigma - i\delta\varphi_\sigma = \delta q_\infty\,(\beta_2\,W^{-3/2} + o(W^{-1})) + \delta a_1\,(W^{-1/2} + o(1)) \tag{17}$$

where β_1, β_2 are given constants, $W = \theta + i\beta_1(\sigma - \sigma_\infty)$, and δa_1 is to be determined.

The perturbed shock is assumed for convenience to be given by $x = X(y) + \delta X(y)$ where $x = X(y)$ describes the unperturbed shock. Expanding the shock conditions one easily finds the perturbed shock conditions on the unperturbed shock curve:

$$[\delta\varphi + u\delta X] = [\delta\psi + \rho v\,\delta X] = 0 \text{ across } x = X(y) \tag{18}$$

where u, v, ρ are the unperturbed velocities and density and [ ] is jump. Thus, since $[\delta X] = 0$, we have $[\delta\varphi][\rho v] + [\delta\psi][u] = 0$ or using (5), as the shock is nearly sonic, we have

$$[\delta\psi] = \sigma_* q_* A_1 \sqrt{\widetilde{\delta q}}\,[\delta\varphi] \tag{19}$$

where $\widetilde{\delta q} = \frac{1}{2}(q_1 + q_2 - 2c_*)$ for the undisturbed shock is proportional to $\sigma_1 + \sigma_2$.

An argument similar to that for the undisturbed shock, § 3, shows that this problem, (14), (16), (17), (19), along with the analogue of (11), counts properly.

A further indication that this problem is correctly posed is given by the following uniqueness theorem where the underlying conditions of the boundary value problem have been preserved.

THEOREM: *Let $\mathfrak{D}$ be a domain bounded by the line $\mathfrak{L}: \theta = 0$, and a convex curve $\mathfrak{C}$. For $\sigma < 0$, the Frankl condition $\sigma\, d\theta^2 + d\sigma^2 < 0$ holds on $\mathfrak{C}$ and also $d\sigma < 0$ (counter clockwise). Suppose u satisfies $\sigma u_{\theta\theta} + u_{\sigma\sigma} = 0$ in $\mathfrak{D}$, $u = 0$ on $\mathfrak{C}$, and $u_\theta(\sigma, 0) = u_\theta(-\sigma, 0)$ holds on $\mathfrak{L}$.*

If u has piecewise continuous second derivatives in $\mathfrak{D}$ and continuous first derivatives in $\overline{\mathfrak{D}}$ then $u \equiv 0$.

Proof: The function $\chi = \int((\sigma u_\theta^2 - u_\sigma^2)\, d\sigma - 2\, u_\theta\, u_\sigma\, d\theta)$ is path independent and assumes its maximum in $\overline{\mathfrak{D}}$ on $\partial\mathfrak{D}$, see [1]. It also satisfies Hopf's strong maximum principle for $\sigma > 0$. By substituting the boundary condition on $\mathfrak{C}$ and noting the conditions on $\mathfrak{C}$ one finds that $d\chi$ can change sign only at the point on $\mathfrak{C}$ where $d\sigma/ds = 0$, $\sigma > 0$. However, at that point, (σ_1, θ_1), $\partial\chi/\partial\sigma \leq 0$. Therefore by Hopf's principle if $\chi \not\equiv$ const. for $\sigma > 0$, its maximum cannot be assumed at (σ_1, θ_1) but must be assumed on $\mathfrak{L}: \theta = 0$, say at σ_2. Thus $\chi(\sigma_1, \theta_1) < \chi(\sigma_2, 0)$. But $\chi(\sigma_2, 0) - \chi(-\sigma_2, 0) = \int_{-\sigma_2}^{\sigma_2}(\sigma u_\theta^2 - u_\sigma^2)\, d\sigma = -\int_{-\sigma_2}^{\sigma_2} u_\sigma^2\, d\sigma$ by the difference boundary condition on $\mathfrak{L}$. But $\chi_\sigma \leq 0$ for $\sigma \leq 0$ and hence $\chi(-\sigma_2, 0) \leq \chi(\sigma_3, 0)$, $\sigma_3 \leq 0$ where $(\sigma_3, 0)$ lies on $\mathfrak{C}$. However, as we have seen, χ is non-decreasing going counterclockwise on $\mathcal{C}$ from $(\sigma_3, 0)$ to (σ_1, θ_1). Hence we have the contradiction,

$$\chi(\sigma_1, \theta_1) < \chi(\sigma_3, 0) \leq \chi(\sigma_1, \theta_1).$$

Thus $\chi \equiv$ constant for $\sigma \geq 0$ and by continuation for $\sigma < 0$. Thus $u \equiv 0$.

This proof could be modified so that either boundary condition $u = 0$ or $\sigma u_\theta \, d\sigma - u_\sigma \, d\theta = 0$ can be imposed on C.

§ 5. SPEED PERTURBATION PROBLEM FOR SMOOTH TRANSONIC FLOW

To formulate this problem we consider it as a limiting case of the preceding problem. As the undisturbed shock conditions we then obtain $[\delta\psi] = 0$ where now [ ] indicates the jump across the sonic boundary point. But from the boundary condition (16) this condition will automatically be satisfied. Thus we seek a solution of (14), (16), (17) with a singularity at the downstream sonic point.

However, the singularity must be weak enough not only to make the problem well posed but so that the disturbance in the physical plane is finite. From $d\varphi + i\rho^{-1} d\psi = W \, dz$ see that $\delta\varphi$, $\delta\psi$ must be continuous.

In addition to describing the development of the shock at the sonic point one must introduce a change of variables in the physical plane. That corresponds to a stretching and makes the shock length finite. The corresponding flow should then behave at infinity as the perturbation flow behaves near the sonic boundary point.

At first glance, there appears to be a contradiction with the "gap" problem described in [2], [3]. But these only show that (14), (16), (17) do not provide a well-posed problem without the admission of singularities. On the contrary, a strong indication that this problem is correctly posed is given by an appropriate theorem for a singular Dirichlet problem for the Tricomi equation [4]. The object there was to find some closed domain for which the Dirichlet problem with data prescribed on the whole closed boundary would be well posed if the solution had its derivatives in an appropriate space with corresponding norm.

The non-homogeneous Tricomi equation, say, $yu_{xx} + u_{yy} = f$, is first replaced by a system for $U = (u_1, u_2)$:

$$yu_{1x} - u_{2y} = f_1 \quad , \quad u_{1y} + u_{2x} = 0$$

which we write as $LU = F$. After formulating a weak existence theorem with the boundary conditions $u_1 \, dx + u_2 \, dy = 0$ we ascertain the most general norm in which existence could be established by the projection theorem. This imposes differential and boundary inequalities on an auxiliary matrix. We determine such a matrix by elementary polynomials and a corresponding domain. However, the resulting norm, although it is locally L^2 away from $y = 0$, admits too wide a class of functions at the parabolic boundary points and the norm must be adjusted to restrict the functions and achieve uniqueness.

Consider the domain D illustrated in fig. 2. Taken counterclockwise ∂D satisfies

(*i*) $x_0 \leq x \leq 0$;

(*ii*) $\left(-x^2 + \frac{4}{9} y^3\right) \mathrm{d}y + \frac{4}{3} x y^{3/2} \mathrm{d}x \geq 0 , y > 0$;

(*iii*) $|x|^{-1} (x \,\mathrm{d}y - y \,\mathrm{d}x) + |x|^{-2} y^3 \left(\frac{1}{3} \mathrm{d}y + \alpha \,\mathrm{d}x\right) > 0 , y < 0$, some $\alpha > 0$;

(*iv*) $|y| < k |x| , |y| < k |x - x_0|$,

(*v*) $y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 > 0 , y < 0$.

The last condition is the Frankl condition that the boundary nowhere becomes characteristic.

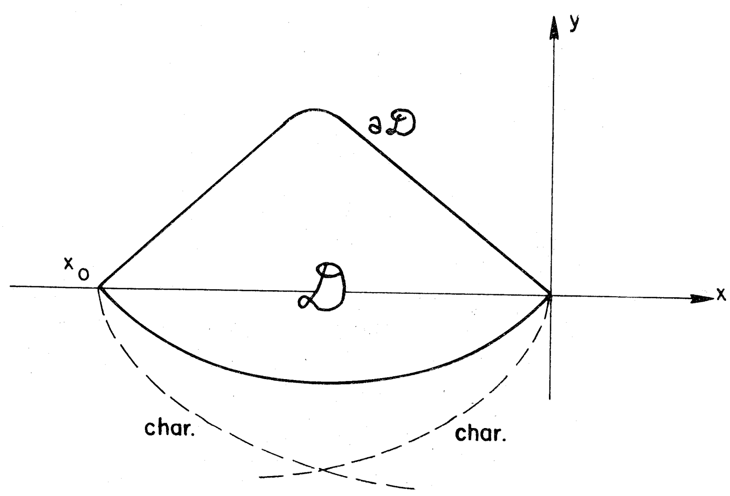

Fig. 2.

Let the positive function

$$p_1 = k |x| y + \varepsilon S(x) \left((x - x_0)^2 + \frac{4}{9} y^3\right)^{-1} y , \qquad y > 0$$

$$= k |y|^3 + \varepsilon S(x) (x - x_0)^{-2} |y| , \qquad y < 0$$

$$p_2 = k |x| + \varepsilon S(x) \left((x - x_0)^2 + \frac{4}{9} y^3\right)^{-1} , \qquad y > 0$$

$$= k |x| + \varepsilon S(x) (x - x_0)^{-2} , \qquad y < 0$$

be used to define the matrix $P = \begin{pmatrix} \sqrt{p_1} & 0 \\ 0 & \sqrt{p_2} \end{pmatrix}$ and let the matrix elements

V_{ik} of V satisfy $v_{11} = v_{22}$, $v_{21} = -yv_{12}$ with

$$v_{11} = -x^2 + \frac{4}{9}y^3 - \varepsilon\left((x-x_0)^2 + \frac{4}{9}y^3\right)^{-1/2} S(x), \qquad y \geq 0$$

$$= -x^2 + \frac{1}{3}y^3 - \varepsilon (x-x_0)^{-1} S(x), \qquad y < 0$$

$$v_{12} = -2xy + \varepsilon \frac{2}{3} y^{3/2}\left((x-x_0)^2 + \frac{4}{9}y^3\right)^{-1} S(x), \qquad y \geq 0$$

$$= -xy - \alpha y^3, \qquad y < 0.$$

Here $S(x)$ represents a positive function, with $S_x \leq 0$, $S \equiv 1$ in a neighborhood of $(x_0, 0)$, $S \equiv 0$ outside a larger neighborhood and smooth in between. The constant $\varepsilon > 0$ is chosen sufficiently small, α is given by (*iii*), and k is an appropriate positive constant.

We define the scalar product $(\cdot,\cdot)$ by integration over $\mathfrak{D}$ and adjoint by $*$.

THEOREM: *If* F *satisfies* $(P^{-1}V^*F, P^{-1}V^*F) < \infty$, *there exists a unique weak solution of* LU=F *satisfying* $(PU, PU) < \infty$ *and* $\int (u_1\,dx + u_2\,dy)^2\,ds = 0$ *on every segment of* $\partial\mathfrak{D}$, $|y| > \delta$. *The solution* U *is in* L^2 *on every space-like curve not intersecting* $y = 0$.

The proof is based on manipulating the inequality $(P\Phi, P\Phi) \leq \leq 2\,|(L\Phi, V\Phi)|$ for all smooth functions Φ satisfying the boundary conditions $\Phi_1\,dx + \Phi_2\,dy = 0$. This inequality comes by straightforward integration by parts. The underlying principle for obtaining from this inequality the weak existence theorem by projection is described in [1], or more generally in [8]. The uniqueness theorem would follow if the solution were smooth and the inequality could be applied directly. Instead a careful smoothing process shows that the inequality holds except for possible contributions from the two parabolic boundary points. These contributions can be shown to be negligible provided the constant ε is not zero.

We note finally that the norm defined in the theorem admits $u_1 = \frac{\partial F}{\partial x}$, $u_2 = \frac{\partial F}{\partial y}$ where F is the fundamental solution of the Tricomi equation, see [7], which behaves near $(0, 0)$ like $r^{-1/3}$ where $r^2 = x^2 + \frac{4}{9}y^3$. This function does not lie in L^2 nor does it lie in the natural norm $\int (|y|\,u_1^2 + u_2^2)\,dx\,dy$ that is associated with Tricomi equation. This suggests that what we have in our theorem is not quite the analogue of the elliptic Dirichlet problem but the analogue of a Dirichlet problem with the solution, prescribed on a closed boundary, admitting a logarithmic singularity at one boundary point and being required to vanish at another.

References

[1] C. S. Morawetz, *Mixed equations and transonic flow*, « Rendiconti di Matematica », *25*, 1–28 (1966).

[2] C. S. Morawetz, *On the nonexistence of continuous transonic flow past profiles*, « Comm. Pure Appl. Math. », I, *9*, 45–68 (1958); II, *10*, 107–131 (1957); III, *11*, 129–144 (1958).

[3] C. S. Morawetz, *Nonexistence of transonic flow past a profile*, « Comm. Pure Appl. Math. », *17*, 357–367 (1964).

[4] C. S. Morawetz, *The Dirichlet problem for the Tricomi equations*, « Comm. Pure Appl. Math. », XXIII, *4* (1970).

[5] P. R. Garabedian and D. Korn, *Numerical design of transonic airfoils*, to appear.

[6] F. I. Frankl, *On the formation of shock waves in subsonic flows with local supersonic velocities*, « Akad, Nauk SSSR, Prikl. Mat. Mech. », *11*, 199–202 (1947).

[7] P. Germain, *Sur quelques problèmes relatifs a l'equation de type mixte de Tricomi*, O.N.E.R.A. No. 54, 1–27 (1952).

[8] K. O. Friedrichs, *Symmetric positive linear differential equations*, « Comm. Pure Appl. Math. », *11*, 333–418 (1958).