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The commutation formulae involving partial derivatives and projective covariant derivatives in a Finsler space

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Geometria differenziale. — *The commutation formulae involving partial derivatives and projective covariant derivatives in a Finsler space.* Nota di K. B. LAL e S. S. SINGH, presentata [(*)] dal Socio E. BOMPIANI.

RIASSUNTO. — In uno spazio di Finsler sono state stabilite formule di permutazione per derivate covarianti secondo Berwald e Cartan [1][(1)] da Sinha [2] e Mishra [3]. In questo lavoro, seguendo lo stesso metodo sono date formule di permutazione per derivate parziali relative all'argomento direzionale e derivate covarianti proiettive. È anche studiato l'effetto di trasformazioni conformi per varie formule di permutazione.

1. INTRODUCTION.—Douglas [4] deduced the projective invariants

$$\Pi^i_{jk}(x\,,\dot{x}) \overset{\text{def}}{=} G^i_{jk} - \frac{1}{n+1}\{2\,\delta^i_{(j}\,G^{\nu}_{k)\nu} + \dot{x}^i\,G^{\nu}_{\nu jk}\}. \tag{1.1}$$

These entities are called the coefficients of projective connection. They are symmetric in their lower indices and are homogeneous of degree zero in their directional arguments. Let the vector $\xi^j(x)$ be taken as the direction of the element of support. The projective covariant derivative of the vector $X^i(x^j, \xi^j)$ with respect to x^k in the direction of $\dot{x}^i$ is given by [6]

(1.2) [(2)]
$$X^i_{((k))}(x\,,\xi) = \partial_k X^i - (\dot{\partial}_m X^i)\,\Pi^m_{pk}\,\xi^p + X^m\,\Pi^i_{mk}.$$

2. THE COMMUTATION FORMULAE INVOLVING THE PARTIAL DERIVATIVES WITH RESPECT TO $\dot{x}^i$ AND PROJECTIVE COVARIANT DERIVATIVES.

THEOREM 2.1.—*The commutation formula for the contravariant vector-field $X^i(x^k, \xi^k)$ which depends on the position and has the direction of the vector $\xi^k(x^h)$ is given by*

(2.1) *a*
$$\dot{\partial}_h X^i_{((k))} - (\dot{\partial}_h X^i)_{((k))} = (\dot{\partial}_m X^i)\,\Pi^m_{hk} - (\dot{\partial}_m X^i)\,\dot{\partial}_h\,\Pi^m_{pk}\,\xi^p + X^m\,\dot{\partial}_h\,\Pi^i_{mk}.$$

Proof.—Differentiating (1.2) with respect to $\dot{x}^h$ we get

$$\dot{\partial}_h X^i_{((k))} = \dot{\partial}_h\,\partial_k X^i - (\dot{\partial}_h\,\dot{\partial}_m X^i)\,\Pi^m_{pk}\,\xi^p - (\dot{\partial}_m X^i)\,\dot{\partial}_h\,\Pi^m_{pk}\,\xi^p + (\dot{\partial}_h X^m)\,\Pi^i_{mk} + X^m\,\dot{\partial}_h\,\Pi^i_{mk}. \tag{2.2}$$

(*) Nella seduta del 14 novembre 1970.

(1) The numbers in the square brackets refer to the references given in the end.

(2) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial\dot{x}^i$.

Also we have

$$(2.3)\qquad (\dot{\partial}_h X^i)_{((k))} = \partial_k \dot{\partial}_h X^i - (\dot{\partial}_m \dot{\partial}_h X^i)\, \Pi^m_{pk}\, \xi^p + (\dot{\partial}_h X^m)\, \Pi^i_{mk} - (\dot{\partial}_m X^i)\, \Pi^m_{hk}.$$

Subtracting (2.3) from (2.2) we get the required formula.

COROLLARY.—*For the covariant vector-field* $B_j(x, \xi)$ *the commutation formulae* (2.1)*a correspond to*

$$(2.1)\,b\qquad \dot{\partial}_h B_{j((k))} - (\dot{\partial}_h B_j)_{((k))} = (\dot{\partial}_m B_j)\, \Pi^m_{hk} - (\dot{\partial}_m B_j)\, \dot{\partial}_h \Pi^m_{pk}\, \xi^p - B_m\, \dot{\partial}_h \Pi^m_{jk}.$$

Proof.—The pattern of proof of this corollary is the same as that of theorem (2.1).

THEOREM 2.2.—*The commutation formula for a contravariant tensor of order two is given by*

$$(2.4)\qquad \dot{\partial}_h T^{ij}{}_{((k))} - (\dot{\partial}_h T^{ij})_{((k))} = (\dot{\partial}_m T^{ij})\, \Pi^m_{hk} - \\ - (\dot{\partial}_m T^{ij})\, \dot{\partial}_h \Pi^m_{pk}\, \xi^p + T^{mj}\, \dot{\partial}_h \Pi^i_{mk} + T^{im}\, \dot{\partial}_h \Pi^j_{mk}.$$

Proof.—Let $B_j(x, \xi)$ be an ordinary covariant vector-field such that its inner product with the tensor $T^{ij}(x, \xi)$ is given by

$$(2.5)\qquad X^i(x, \xi) \stackrel{\text{def}}{=} T^{ij}(x, \xi)\, B_j(x, \xi).$$

Eliminating $X^i(x, \xi)$ from (2.1) *a* and (2.5) and using (2.1) *b* we obtain

$$B_j \{ \dot{\partial}_h T^{ij}{}_{((k))} - (\dot{\partial}_h T^{ij})_{((k))} - (\dot{\partial}_m T^{ij})\, \Pi^m_{hk} + (\dot{\partial}_m T^{ij})\, \dot{\partial}_h \Pi^m_{pk}\, \xi^p - \\ - T^{mj}\, \dot{\partial}_h \Pi^i_{mk} - T^{im}\, \dot{\partial}_h \Pi^j_{mk} \} = 0.$$

Since B_j is arbitrary, (2.4) follows from above equation.

THEOREM 2.3.—*The commutation formula for the contravariant tensor* $T^{i_1 \cdots i_p}(x, \xi)$ is given *by*

$$(2.6)\qquad \dot{\partial}_h T^{i_1 \cdots i_p}{}_{((k))} - (\dot{\partial}_h T^{i_1 \cdots i_p})_{((k))} = (\dot{\partial}_\nu T^{i_1 \cdots i_p})\, \Pi^\nu_{hk} - \\ - (\dot{\partial}_\nu T^{i_1 \cdots i_p})\, \dot{\partial}_h \Pi^\nu_{pk}\, \xi^p + \sum_{\alpha=1}^{p} T^{i_1 \cdots i_{\alpha-1} \mu i_{\alpha+1} \cdots i_p}\, \dot{\partial}_h \Pi^{i_\alpha}_{\mu k}.$$

Proof.—Let the theorem be true for a contravariant tensor of order $m\,(< p)$. Thus, we have

$$(2.7)\qquad \dot{\partial}_h X^{i_1 \cdots i_m}{}_{((k))} - (\dot{\partial}_h X^{i_1 \cdots i_m})_{((k))} = (\dot{\partial}_\nu X^{i_1 \cdots i_m})\, \Pi^\nu_{hk} - \\ - (\dot{\partial}_\nu X^{i_1 \cdots i_m})\, \dot{\partial}_h \Pi^\nu_{hk}\, \xi^p + \sum_{\alpha=1}^{m} X^{i_1 \cdots i_{\alpha-1} \mu i_{\alpha+1} \cdots i_m}\, \dot{\partial}_h \Pi^{i_\alpha}_{\mu k}.$$

Let the inner product of an $(m+1)$th order contravariant tensor $T^{i_1\cdots i_m\lambda}(x,\xi)$ with an arbitrary covariant vector-field $B_\lambda(x,\xi)$ be given by

$$X^{i_1\cdots i_m}(x,\xi) \overset{\text{def}}{=} T^{i_1\cdots i_m\lambda}(x,\xi)\,B_\lambda(x,\xi)\,. \tag{2.8}$$

Eliminating $X^{i_1\cdots i_m}(x,\xi)$ from (2.7) and (2.8) and using (2.1) b we get

$$B_\lambda\Big[\dot\partial_h T^{i_1\cdots i_m\lambda}{}_{((k))} - (\dot\partial_h T^{i_1\cdots i_m\lambda})_{((k))} - (\dot\partial_\nu T^{i_1\cdots i_m\lambda})\,\Pi^{\nu}_{hk} +$$
$$+ (\dot\partial_\nu T^{i_1\cdots i_m\lambda})\,\dot\partial_h \Pi^{\nu}_{pk}\,\xi^p - \sum_{\alpha=1}^{m} T^{i_1\cdots i_{\alpha-1}\mu i_{\alpha+1}\cdots i_m\lambda}\,\dot\partial_h \Pi^{i_\alpha}_{\mu k} - T^{i_1\cdots i_m\mu}\,\dot\partial_h \Pi^{\lambda}_{\mu k}\Big] = 0\,.$$

Since B_λ is an arbitrary vector-field, replacing the index λ by i_{m+1} in the above equation we obtain

$$\dot\partial_h T^{i_1\cdots i_{m+1}}{}_{((k))} - (\dot\partial_h T^{i_1\cdots i_{m+1}})_{((k))} = (\dot\partial_\nu T^{i_1\cdots i_{m+1}})\,\Pi^{\nu}_{hk} - \tag{2.9}$$
$$- (\dot\partial_\nu T^{i_1\cdots i_{m+1}})\,\dot\partial_h \Pi^{\nu}_{sk}\,\xi^s + \sum_{\alpha=1}^{m+1} T^{i_1\cdots i_{\alpha-1}\mu i_{\alpha+1}\cdots i_{m+1}}\,\dot\partial_h \Pi^{i_\alpha}_{\mu k}\,.$$

This equation is similar to (2.7). Hence if (2.6) is true for a contravariant tensor of order m, it is true for that of order $(m+1)$. In Theorem (2.2) it has been already shown that (2.6) is true for a contravariant tensor of order two. Therefore, (2.6) is true for a contravariant tensor of any order p.

THEOREM 2.4.—*The commutation formula for a covariant tensor of order two is given by*

$$\dot\partial_h T_{ij((k))} - (\dot\partial_h T_{ij})_{((k))} = (\dot\partial_m T_{ij})\,\Pi^m_{hk} - (\dot\partial_m T_{ij})\,\dot\partial_h \Pi^m_{pk}\,\xi^p - \tag{2.10}$$
$$- T_{mj}\,\dot\partial_h \Pi^m_{ik} - T_{im}\,\dot\partial_h \Pi^m_{jk}\,.$$

Proof.—Let $X^i(x,\xi)$ be an arbitrary contravariant vector-field whose inner product with the tensor $T_{ij}(x,\xi)$ is given by

$$B_j(x,\xi) \overset{\text{def}}{=} T_{ij}(x,\xi)\,X^i(x,\xi). \tag{2.11}$$

Eliminating B_j from (2.1) b and (2.11) and using (2.1) a we get

$$X^i\,[\dot\partial_h T_{ij((k))} - (\dot\partial_h T_{ij})_{((k))} - (\dot\partial_m T_{ij})\,\Pi^m_{hk} + (\dot\partial_m T_{ij})\,\dot\partial_h \Pi^m_{pk}\,\xi^p +$$
$$+ T_{mj}\,\dot\partial_h \Pi^m_{ik} + T_{im}\,\dot\partial_h \Pi^m_{jk}] = 0\,.$$

Since X^i is an arbitrary vector-field we have the formula (2.10).

THEOREM 2.5.—*The commutation formula for a covariant tensor of arbitrary rank q is given by*

$$(2.12)\qquad \dot{\partial}_h T_{j_1\cdots j_q((k))} - (\dot{\partial}_h T_{j_1\cdots j_q})_{((k))} = (\dot{\partial}_\nu T_{j_1\cdots j_q})\, \Pi^\nu_{hk} - $$

$$- (\dot{\partial}_\nu T_{j_1\cdots j_q})\, \dot{\partial}_h \Pi^\nu_{pk}\, \xi^p - \sum_{\beta=1}^{q} T_{j_1\cdots j_{\beta-1}\mu j_{\beta+1}\cdots j_q}\, \dot{\partial}_h \Pi^\mu_{j_\beta k}\,.$$

Proof.—Let the theorem be true for a covariant tensor of order $m\ (< q)$. Thus, we have

$$(2.13)\qquad \dot{\partial}_h B_{j_1\cdots j_m((k))} - (\dot{\partial}_h B_{j_1\cdots j_m})_{((k))} = (\dot{\partial}_\nu B_{j_1\cdots j_m})\, \Pi^\nu_{hk} - $$

$$- (\dot{\partial}_\nu B_{j_1\cdots j_m})\, \dot{\partial}_h \Pi^\nu_{pk}\, \xi^p - \sum_{\beta=1}^{m} B_{j_1\cdots j_{\beta-1}\mu j_{\beta+1}\cdots j_m}\, \dot{\partial}_h \Pi^\mu_{j_\beta k}\,.$$

Let the inner product of an $(m+1)$th order covariant tensor $T_{j_1\cdots j_m\lambda}(x, \xi)$ with an arbitrary contravariant vector-field $X^\lambda(x, \xi)$ be defined by

$$(2.14)\qquad B_{j_1\cdots j_m}(x, \xi) \overset{\text{def}}{=} T_{j_1\cdots j_m\lambda}(x, \xi)\, X^\lambda(x, \xi)\,.$$

Substituting the value of $B_{j_1\cdots j_m}(x, \xi)$ obtained from (2.14) in (2.13) we obtain in view of (2.1) *a*

$$X^\lambda \Big[\dot{\partial}_h T_{j_1\cdots j_m\lambda((k))} - (\dot{\partial}_h T_{j_1\cdots j_m\lambda})_{((k))} - (\dot{\partial}_\nu T_{j_1\cdots j_m\lambda})\, \Pi^\nu_{hk} + $$

$$+ (\dot{\partial}_\nu T_{j_1\cdots j_m\lambda})\, \dot{\partial}_h \Pi^\nu_{pk}\, \xi^p + \sum_{\beta=1}^{m} T_{j_1\cdots j_{\beta-1}\mu j_{\beta+1}\cdots j_m\lambda}\, \dot{\partial}_h \Pi^\mu_{j_\beta k} + T_{j_1\cdots j_m\mu}\, \dot{\partial}_h \Pi^\mu_{\lambda k} \Big] = 0\,.$$

Since X^λ is an arbitrary vector-field, replacing the index λ by j_{m+1} in the above equation, we get

$$(2.15)\qquad \dot{\partial}_h T_{j_1\cdots j_{m+1}((k))} - (\dot{\partial}_h T_{j_1\cdots j_{m+1}})_{((k))} = (\dot{\partial}_\nu T_{j_1\cdots j_{m+1}})\, \Pi^\nu_{hk} - $$

$$- (\dot{\partial}_\nu T_{j_1\cdots j_{m+1}})\, \dot{\partial}_h \Pi^\nu_{pk}\, \xi^p - \sum_{\beta=1}^{m+1} T_{j_1\cdots j_{\beta-1}\mu j_{\beta+1}\cdots j_{m+1}}\, \dot{\partial}_h \Pi^\mu_{j_\beta k}\,.$$

Thus the formula is true also for a covariant tensor of order $(m+1)$. It has been already shown to be true for a covariant tensor of order two. Hence it is true for a covariant tensor of any order q.

THEOREM 2.6.—*The commutation formula for a mixed tensor of contravariant order p and covariant order q is given by*

$$(2.16)\qquad \dot{\partial}_h T^{i_1\cdots i_p}_{j_1\cdots j_q((k))} - (\dot{\partial}_h T^{i_1\cdots i_p}_{j_1\cdots j_q})_{((k))} = (\dot{\partial}_\nu T^{i_1\cdots i_p}_{j_1\cdots j_q})\, \Pi^{\nu}_{hk} - (\dot{\partial}_\nu T^{i_1\cdots i_p}_{j_1\cdots j_q})\, \dot{\partial}_h \Pi^{\nu}_{sk}\, \xi^s + \sum_{\alpha=1}^{p} T^{i_1\cdots i_{\alpha-1}\mu i_{\alpha+1}\cdots i_p}_{j_1\cdots\cdots\cdots j_q}\, \dot{\partial}_h \Pi^{i_\alpha}_{\mu k} - \sum_{\beta=1}^{q} T^{i_1\cdots\cdots\cdots i_p}_{j_1\cdots j_{\beta-1}\mu i_{\beta+1}\cdots j_q}\, \dot{\partial}_h \Pi^{\mu}_{j_\beta k}\,.$$

Proof.—The proof of this theorem follows from that of theorems (2.3) and (2.5).

3. CONFORMAL FINSLER SPACE.—Let two distinct metric functions $F(x, \dot{x})$ and $\bar{F}(x, \dot{x})$ be defined over an n–dimensional Finsler space which satisfy the requisite conditions for a Finsler space. The two metrics resulting from these functions are called conformal if the corresponding metric tensors $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$ are proportional to each other. It has been proved that the factor of proportionality between them is at most a point function. Thus we have [1]

$$(3.1)\quad \bar{g}_{ij}(x, \dot{x}) = e^{2\sigma} g_{ij}(x, \dot{x}), \qquad (3.2)\quad \bar{g}^{ij}(x, \dot{x}) = e^{-2\sigma} g^{ij}(x, \dot{x}),$$

and

$$(3.3)\qquad \bar{F}(x, \dot{x}) = e^{\sigma} F(x, \dot{x}).$$

where $\sigma = \sigma(x)$, g^{ij} being the contravariant components of the metric tensor of F_n. The space $\bar{F}_n$ with the entities $\bar{F}, \bar{g}_{ij}$, etc. is called a conformal Finsler space. We shall use the following geometric entities in the conformal Finsler space given by [5]

$$(3.4)\qquad \bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - \sigma_m B^{im},$$

$$(3.5)\qquad \bar{G}^i_{jh}(x, \dot{x}) = G^i_{jh}(x, \dot{x}) - \sigma_m \dot{\partial}_h \dot{\partial}_j B^{im},$$

$$(3.6)\,a\quad \bar{C}^i_{jh}(x, \dot{x}) = C^i_{jh}(x, \dot{x}), \qquad (3.6)\,b\quad \bar{C}_{ijh}(x, \dot{x}) = e^{2\sigma} C_{ijh}(x, \dot{x}),$$

$$(3.6)\,c\quad \bar{\dot{x}}^i = \dot{x}^i, \qquad (3.6)\,d\quad \bar{l}^i(x, \dot{x}) = e^{-\sigma} l^i(x, \dot{x}),$$

$$(3.6)\,e\quad \bar{l}_i(x, \dot{x}) = e^{\sigma} l_i(x, \dot{x})$$

where

$$(3.7)\,a\quad \bar{C}_{ijh}(x, \dot{x}) \stackrel{\text{def}}{=} \dot{\partial}_h \bar{g}_{ij}(x, \dot{x}), \qquad (3.7)\,b\quad \bar{C}^l_{ih} \stackrel{\text{def}}{=} \bar{g}^{lj}(x, \dot{x})\, \bar{C}_{ijh}(x, \dot{x}),$$

$$(3.7)\,c\quad B^{hk}(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2} F^2 g^{hk} - \dot{x}^h \dot{x}^k, \qquad (3.7)\,d\quad \sigma_m \stackrel{\text{def}}{=} \partial_m \sigma.$$

Also we have [6]

$$\overline{\Pi}^{i}_{lk}(x, \dot{x}) = \Pi^{i}_{lk}(x, \dot{x}). \tag{3.8}$$

Where $\Pi^{i}_{jk}(x, \dot{x})$ is given by (1.1).

4. THE COMMUTATION FORMULAE IN CONFORMAL FINSLER SPACE ANALOGOUS TO (2.1).

THEOREM 4.1.—*When* $F_n(x, \dot{x})$ *and* $\overline{F}_n(x, \dot{x})$ *are in conformal correspondence, the commutation formulae for* $\overline{F}$ *and the vector* ($\overline{l}^i$ *and* l_i) *in the direction of the element of support are given by*

$$\dot{\partial}_h \overline{F}_{\overline{((k))}} - (\dot{\partial}_h \overline{F})_{\overline{((k))}} = e^{\sigma}\{\dot{\partial}_h F_{((k))} - (\dot{\partial}_h F)_{((k))}\}, \tag{4.1 a}$$

$$\dot{\partial}_h \overline{l}^i_{\overline{((k))}} - (\dot{\partial}_h \overline{l}^i)_{\overline{((k))}} = e^{-\sigma}\{\dot{\partial}_h l^i_{((k))} - (\dot{\partial}_h l^i)_{((k))}\}, \tag{4.1 b}$$

and

$$\dot{\partial}_h \overline{l}_{i\overline{((k))}} - (\dot{\partial}_h \overline{l}_i)_{\overline{((k))}} = e^{\sigma}\{\dot{\partial}_h l_{i((k))} - (\dot{\partial}_h l_i)_{((k))}\}, \tag{4.1 c}$$

where the notation $\overline{((k))}$ *denotes the projective covariant derivative for the connection coefficients* $\overline{\Pi}^{i}_{jk}(x, \dot{x})$ *in conformal Finsler space (i.e.* $\overline{X}^i_{\overline{((k))}} = (\partial_k \overline{X}^i) - (\dot{\partial}_m \overline{X}^i)\, \overline{\Pi}^m_{pk}\, \dot{x}^p + \overline{X}^m\, \overline{\Pi}^i_{mk}$).

Proof.—The commutation formula (2.1) in conformal Finsler space for $\overline{F}$ will be given by

$$\dot{\partial}_h \overline{F}_{\overline{((k))}} - (\dot{\partial}_h \overline{F})_{\overline{((k))}} = (\dot{\partial}_m \overline{F})\, \overline{\Pi}^m_{hk} - (\dot{\partial}_m \overline{F})\, \dot{\partial}_h \overline{\Pi}^m_{pk}\, \xi^p. \tag{4.2}$$

Using equations (3.3), (3.6) *c* and (3.8) we get (4.1) *a*. Also the commutation formula (2.1) in conformal Finsler space for the vector $\overline{l}^i(x, \dot{x})$ will be given by

$$\dot{\partial}_h \overline{l}^i_{\overline{((k))}} - (\dot{\partial}_h \overline{l}^i)_{\overline{((k))}} = (\dot{\partial}_m \overline{l}^i)\, \overline{\Pi}^m_{hk} - (\dot{\partial}_m \overline{l}^i)\, \dot{\partial}_h \overline{\Pi}^m_{pk}\, \xi^p + \overline{l}^m \dot{\partial}_h \overline{\Pi}^i_{mk}. \tag{4.3}$$

Using equations (3.6) *c*, (3.6) *d* and (3.8), we obtain (4.1) *b*. Similarly, proceeding on the same lines as above, for the covariant component of the vector $\overline{l}_i(x, \dot{x})$ we get (4.1) *c*.

THEOREM 4.2.—*When* $F_n(x, \dot{x})$ *and* $\overline{F}_n(x, \dot{x})$ *are in conformal correspondence, we have*

$$\dot{\partial}_h \overline{g}_{ij\overline{((k))}} - (\dot{\partial}_h \overline{g}_{ij})_{\overline{((k))}} = e^{2\sigma}\{\dot{\partial}_h g_{ij((k))} - (\dot{\partial}_h g_{ij})_{((k))}\}, \tag{4.3 a}$$

$$\dot{\partial}_h \overline{g}^{ij}_{\overline{((k))}} - (\dot{\partial}_h \overline{g}^{ij})_{\overline{((k))}} = e^{-2\sigma}\{\dot{\partial}_h g^{ij}_{((k))} - (\dot{\partial}_h g^{ij})_{((k))}\}, \tag{4.3 b}$$

$$\dot{\partial}_h \overline{C}^m_{ij\overline{((k))}} - (\dot{\partial}_h \overline{C}^m_{ij})_{\overline{((k))}} = \{\dot{\partial}_h C^m_{ij((k))} - (\dot{\partial}_h C^m_{ij})_{((k))}\}, \tag{4.3 c}$$

and

$$(4.3)\,d \qquad \dot{\partial}_h \bar{C}_{ijm\overline{((k))}} - (\dot{\partial}_h \bar{C}_{ijm})_{\overline{((k))}} = e^{2\sigma}\{\dot{\partial}_h C_{ijm((k))} - (\dot{\partial}_h C_{ijm})_{((k))}\}.$$

Proof.—The pattern of proof of theorem (4.2) is the same as that of the theorem (4.1).

THEOREM 4.3.—*When* $F_n(x, \dot{x})$ *and* $\bar{F}_n(x, \dot{x})$ *are in conformal correspondence the commutation formulae for* $\bar{G}^i(x, \dot{x})$ *and* $\bar{G}^i_{jk}(x, \dot{x})$ *are given by*

$$(4.4)\,a \qquad \dot{\partial}_h \bar{G}^i_{\overline{((k))}} - (\dot{\partial}_h \bar{G}^i)_{\overline{((k))}} = [\{\dot{\partial}_h G^i_{((k))} - (\dot{\partial}_h G^i)_{((k))}\} - \\ - \sigma_n \{\dot{\partial}_h B^{in}{}_{((k))} - (\dot{\partial}_h B^{in})_{((k))} - B^{im}\dot{\partial}_h \Pi^n_{mk}\}],$$

and

$$(4.4)\,b \qquad \dot{\partial}_h \bar{G}^m_{ij\overline{((k))}} - (\dot{\partial}_h \bar{G}^m_{ij})_{\overline{((k))}} = [\{\dot{\partial}_h G^m_{ij((k))} - (\dot{\partial}_h G^m_{ij})_{((k))}\} - \\ - \sigma_n \{\dot{\partial}_h (\dot{\partial}_i \dot{\partial}_j B^{mn})_{((k))} - (\dot{\partial}_h \dot{\partial}_i \dot{\partial}_j B^{mn})_{((k))} - (\dot{\partial}_i \dot{\partial}_j B^{ml})\dot{\partial}_h \Pi^n_{lk}\}].$$

Proof.—The commutation formula (2.1) in conformal Finsler space for $\bar{G}^i(x, \dot{x})$ will read as follows:

$$(4.5) \qquad \dot{\partial}_h \bar{G}^i_{\overline{((k))}} - (\dot{\partial}_h \bar{G}^i)_{\overline{((k))}} = \dot{\partial}_m \bar{G}^i \bar{\Pi}^m_{hk} - (\dot{\partial}_m \bar{G}^i)\dot{\partial}_h \bar{\Pi}^m_{pk}\xi^p + \bar{G}^m \dot{\partial}_h \bar{\Pi}^i_{mk}.$$

With the help of equations (3.4), (3.6) *c* and (3.8) the above equation reduces to the form

$$(4.6) \qquad \dot{\partial}_h \bar{G}^i_{\overline{((k))}} - (\dot{\partial}_h \bar{G}^i)_{\overline{((k))}} = [\{\dot{\partial}_m G^i \Pi^m_{hk} - (\dot{\partial}_m G^i)\dot{\partial}_h \Pi^m_{pk}\xi^p + G^m \dot{\partial}_h \Pi^i_{mk}\} - \\ - \sigma_n \{\dot{\partial}_m B^{in}\Pi^m_{hk} - (\dot{\partial}_m B^{in})\dot{\partial}_h \Pi^m_{pk}\xi^p + B^{mn}\dot{\partial}_h \Pi^i_{mk}\}].$$

By adding and subtracting a term $\sigma_n B^{im}\dot{\partial}_h \Pi^n_{mk}$ we get the result (4.3) *a*.

Again, for the tensor $\bar{G}^i_{jk}(x, \dot{x})$ the commutation formula (2.1) will read as follows:

$$(4.7) \qquad \dot{\partial}_h \bar{G}^m_{ij\overline{((k))}} - (\dot{\partial}_h \bar{G}^m_{ij})_{\overline{((k))}} = \dot{\partial}_l \bar{G}^m_{ij}\bar{\Pi}^l_{hk} - (\dot{\partial}_l \bar{G}^m_{ij})\dot{\partial}_h \bar{\Pi}^l_{pk}\xi^p + \\ + \bar{G}^l_{ij}\dot{\partial}_h \bar{\Pi}^m_{lk} - \bar{G}^m_{lj}\dot{\partial}_h \bar{\Pi}^l_{ik} - \bar{G}^m_{il}\dot{\partial}_h \bar{\Pi}^l_{jk},$$

which gives using equations (3.5), (3.6)*c* and (3.8)

$$(4.8) \qquad \dot{\partial}_h \bar{G}^m_{ij\overline{((k))}} - (\dot{\partial}_h \bar{G}^m_{ij})_{\overline{((k))}} = [\{\dot{\partial}_h G^m_{ij((k))} - (\dot{\partial}_h G^m_{ij})_{((k))}\} - \\ - \sigma_n \{\dot{\partial}_l (\dot{\partial}_i \dot{\partial}_j B^{mn})\Pi^l_{hk} - \dot{\partial}_i (\dot{\partial}_i \dot{\partial}_j B^{mn})\dot{\partial}_h \Pi^l_{pk}\xi^p + \\ + \dot{\partial}_i \dot{\partial}_j B^{ln}\dot{\partial}_h \Pi^m_{lk} - \dot{\partial}_l \dot{\partial}_j B^{mn}\dot{\partial}_h \Pi^l_{ik} - \dot{\partial}_i \dot{\partial}_l B^{mn}\Pi^l_{jk}\}].$$

Now, adding and subtracting a term $\sigma_l(\dot{\partial}_i \dot{\partial}_j B^{ml})\dot{\partial}_h \Pi^n_{lk}$ in (4.8) we obtain the theorem.

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