Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A Lemma on maps of a compact topological space and an application to fixed point theory

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **49** (1970), n.5, p. 242–243.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1970_8_49_5_242_0>

Analisi funzionale. — *A Lemma on maps of a compact topological space and an application to fixed point theory* (*). Nota di MARIO MARTELLI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un lemma sulle trasformazioni, non necessariamente continue, di uno spazio topologico compatto in sè, e se ne dà un'applicazione alla teoria dei punti fissi.

The main result of the present paper is represented by a Lemma on mappings (not necessarily continuous) of a compact topological space into itself. This Lemma seems not to have been noticed before in spite of its simplicity and the possibility of applications to fixed point theory. As an application of this kind we give a new and unique proof of two fixed point theorems due to B. N. Sadovskii [1] and M. Furi–A. Vignoli [2] respectively.

1. LEMMA.—*Let* T *be a mapping of a compact topological space* K *into itself. Then there exists a nonempty subset* $M \subset K$ *such that* $M = \overline{T(M)}$.

Proof. Let $\mathfrak{F}$ be the family of all nonempty closed subset A of K such that $T(A) \subset A$. $\mathfrak{F}$ is nonempty since $K \in \mathfrak{F}$. Moreover $\mathfrak{F}$ can be partially ordered by the set inclusion. Let $\mathfrak{D}$ be a totally ordered subset of $\mathfrak{F}$ and put $Q = \bigcap_{C \in \mathfrak{D}} C$. Q is nonempty by the compactness of K and it is a lower bound of $\mathfrak{D}$. Using Zorn's lemma we can find a minimal set M of $\mathfrak{F}$. Clearly $M = \overline{T(M)}$ by the minimality of M.

Note that if T is continuous and K is Hausdorff space then T(M) is compact, therefore closed. Hence $M = T(M)$.

2. To give an application of the preceding Lemma to fixed point theory we recall first some terminologies.

Let A be a bounded set of a Banach space E. We denote by $\alpha(A)$ [3] the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε and by $\chi(A)$ the infimum of all $\varepsilon > 0$ for which A has a finite ε–net. Clearly $\chi(A) \leq \alpha(A) \leq 2\chi(A)$ and $\alpha(A) = \chi(A) = 0$ iff A is precompact, i.e. totally bounded.

G. Darbo [4] and B. N. Sadovskii [1] proved that 1. $\alpha(A) = \alpha(\overline{co}(A))$, 2. $\chi(A) = \chi(\overline{co}(A))$ respectively, where $\overline{co}(A)$ indicates the convex closure of A.

Let Q be a nonempty subset of E and $T : Q \to E$ a continuous mapping such that $\chi(T(A)) < \chi(A)$ $(\alpha(T(A)) < \alpha(A))$ for any bounded and non

(*) Lavoro eseguito nell'ambito delle attività dei Contratti di Ricerca Matematica del C.N.R.

(**) Nella seduta del 14 novembre 1970.

precompact subset $A \subset Q$. In the first case T is called *condensing* [1], in the second one *densifying* [2]. B. N. Sadovskii [1] proved that a condensing mapping T, which maps a convex closed bounded set A of a Banach space E into itself, has at least one fixed point in A. M. Furi–A. Vignoli [2] proved that the same result holds for a densifying mapping T.

We now give a new unique proof of these two results using Lemma and the properties 1. and 2.

THEOREM (Sadovskii–Furi–Vignoli).—*Let Q be a nonempty closed convex bounded set of a Banach space E and let $T : Q \to Q$ be a condensing (densifying) mapping. Then T has at least one fixed point in Q.*

Proof. Consider the sequence $\{T^n(x_0), n = 0, 1, \cdots\}$ of iterates, starting from x_0, where x_0 is a given point of Q. Since T is a condensing (densifying) mapping, $K = \overline{\{T^n(x_0), n = 0, 1 \ldots\}}$ is invariant and compact. Let M be the subset of K whose existence is insured by the Lemma and consider the family $\mathcal{F} = \{B \subset Q : M \subset B, B$ closed convex invariant under $T\}$. Put $A = \cap \{B : B \in \mathcal{F}\}$. Clearly $A = \overline{co}(T(A))$. Since $\chi(\overline{co}(T(A))) = \chi(T(A))$ $(\alpha(\overline{co}(T(A))) = \alpha(T(A)))$, A is compact. Then, by the Schauder–Tichonov fixed point theorem, there exists $y \in A$ such that $T(y) = y$.

Remark. An extension of the above Theorem can be obtained by replacing the hypothesis "Q is bounded" with "the range of T is bounded" or, more generally, "there exists a bounded sequence of iterates $\{T^n(x_0)\}$, where $x_0 \in Q$".

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