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On some finite structures

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 novembre 1970

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On some finite structures*. Nota di ALESSANDRA GIOVAGNOLI, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Questa Nota, tratta dalla tesi presentata dall'autrice all'Università di Londra per il Master of Philosophy, espone riassuntivamente alcuni risultati relativi a configurazioni tattiche e 2–disegni definiti a partire da polarità di uno spazio di Galois.

We consider polarities of a finite projective space PG (d, q) of dimension d over the field GF (q). Our terminology and notation will be in accordance with [3]. When dealing with orthogonal polarities, it will be implicitly assumed that q be odd. By " quadric " we shall mean the set of absolute points of an orthogonal polarity (see [6]), by " hermitian form " the set of absolute points of a unitary polarity, in which case q must be a square.

The scope of this Note is to give a summary account of some results, most of which are part of the author's thesis for the M. Phil. Degree of London University, about tactical configurations and designs obtained from the set of absolute points of a polarity, and such that their collineation group contains an isomorphic copy of the group of the polarity. Definitions and basic facts about designs can be found in [7].

A more detailed paper will appear subsequently in which all the proofs that are outlined only briefly in the present one will be dealt with more diffusely.

(*) Nella seduta del 14 novembre 1970.

METHOD N. 1

Roughly, this consists of taking absolute points and non-absolute hyperplanes of a polarity. The following structures are obtained:

$D_1 H(2n+1, q)$: whose "points" are those of a hyperbolic quadric in odd dimension $2n+1$ and blocks the non-absolute hyperplanes. Incidence is the same as in $PG(2n+1, q)$.

$D_1 E(2n+1, q)$: ditto with an elliptic quadric instead.

PROPOSITION 1.—*$D_1 H(2n+1, q)$ and $D_1 E(2n+1, q)$ are T.C.'s (short, from now on, for "tactical configurations") with the following parameters*

$$D_1 H(2n+1, q): \begin{cases} v = \dfrac{(q^{n+1}-1)(q^n+1)}{q-1}, \\ b = q^n(q^{n+1}-1), \\ k = \dfrac{q^{2n}-1}{q-1}. \end{cases} \qquad D_1 E(2n+1, q): \begin{cases} v = \dfrac{(q^{n+1}+1)(q^n-1)}{q-1}, \\ b = q^n(q^{n+1}+1), \\ k = \dfrac{q^{2n}-1}{q-1}. \end{cases}$$

The proof is achieved by applying Witt's theorem and Lemma 1 of Chapter IV.4 at p. 112 of [1].

PROPOSITION 2.—*Apart from $D_1 E(3, q)$ which is a 3-design and in fact an inversive plane, none of the remaining structures of the type $D_1 H(2n+1, q)$ or $D_1 E(2n+1, q)$ is a 2-design.*

Proof: By contradiction: we assume one of such structures is a 2-design, work out the would-be parameter λ_2 and, using congruences, show that λ_2 is never an integer.

More incidence structures are obtained considering a quadric $\mathcal{Q}_{2n}$ in even dimension $d = 2n$.

$D_1 H(2n, q)$: whose points are those of $\mathcal{Q}_{2n}$ and whose blocks are hyperplanes of $PG(2n, q)$ intersecting $\mathcal{Q}_{2n}$ in a *hyperbolic* quadric Q_{2n-1}.

$D_1 E(2n, q)$: ditto, with hyperplanes intersecting $\mathcal{Q}_{2n}$ in an *elliptic* $\mathcal{Q}_{2n-1}$.

PROPOSITION 3.—*$D_1 H(2n, q)$ and $D_1 E(2n, q)$ are T.C.'s with the following parameters*

$$D_1 H(2n, q): \begin{cases} v = \dfrac{q^{2n}-1}{q-1}, \\ b = \dfrac{q^n(q^n+1)}{2}, \\ k = \dfrac{(q^n-1)(q^{n-1}+1)}{q-1}. \end{cases} \qquad D_1 E(2n, q): \begin{cases} v = \dfrac{q^{2n}-1}{q-1}, \\ b = \dfrac{q^n(q^n-1)}{2}, \\ k = \dfrac{(q^n+1)(q^{n-1}-1)}{q-1}. \end{cases}$$

The core of this proof lies in disentangling blocks of $D_1 H(2n, q)$ from those of $D_1 E(2n, q)$. The rest is straightforward.

PROPOSITION 4.—$D_1 H(2n, q)$ *and* $D_1 E(2n, q)$ *are never 2-designs.* Proof as Prop. 2.

Now take a hermitian form $\mathcal{H}_d$. An incidence structure $D_1 U(d, q)$ is obtained, whose "points" are the points of $\mathcal{H}_d$ and whose blocks are the non-absolute hyperplanes.

PROPOSITION 5.—$D_1 U(d, q)$ *is a T.C. with parameters as follows*:

Let $n \geq 1$

$$\text{for } d = 2n \begin{cases} v = \dfrac{(q^n \sqrt{q} + 1)(q^n - 1)}{q - 1}, \\[2ex] b = q^n \dfrac{q^n \sqrt{q} + 1}{q + 1}, \\[2ex] k = \dfrac{(q^{n-1} \sqrt{q} + 1)(q^n - 1)}{q - 1}. \end{cases} \qquad \text{for } d = 2n + 1 \begin{cases} v = \dfrac{(q^n \sqrt{q} + 1)(q^{n+1} - 1)}{q - 1}. \\[2ex] b = q^n \dfrac{q^{n+1} - 1}{q + 1}, \\[2ex] k = \dfrac{(q^n \sqrt{q} + 1)(q^n - 1)}{q - 1}. \end{cases}$$

Proof as Prop. 1.

PROPOSITION 6.—$D_1 U(2, q)$ *is a* $2 - (q\sqrt{q} + 1, \sqrt{q} + 1, 1)$ *design. For* $d \geq 3$ $D_1 U(d, q)$ *is never a 2-design.*

The first result stated is well-known. For the proof of the second see [4] p. 31–32.

Let π be a symplectic polarity. Then d must be odd. Taking absolute points and non-absolute hyperplanes clearly leads nowhere. However, we can still find a T.C., $D_1 S(d, q)$, by picking totally isotropic lines as points and non-isotropic $(d-2)$-dimensional subspaces of $PG(d, q)$ as blocks.

PROPOSITION 7.—*For* $d \geq 5$, $D_1 S(d, q)$ *is a T.C. with parameters*:

$$v = \frac{(q^{d+1} - 1)(q^{d-1} - 1)}{(q - 1)(q^2 - 1)}; \quad b = q^{d-1} \frac{q^{d+1} - 1}{q^2 - 1}; \quad k = \frac{(q^{d-1} - 1)(q^{d-3} - 1)}{(q - 1)(q^2 - 1)},$$

which is never a 2-design.

Proof: For the parameters see [3]. The second part is proved similarly to Proposition 2.

METHOD N. 2

Absolute points and hyperplanes of a polarity π with incidence as in $PG(d, q)$ can be taken as "points" and blocks of an incidence structure $D_2(\pi)$. By Witt's theorem, the group of the polarity acts transitively on the blocks as well as on the points of $D_2(\pi)$. Hence it follows:

PROPOSITION 8.—$D_2(\pi)$ *is a symmetric* (*i.e. projective*) *T.C.*

The table of parameters is shown below; the meaning of the symbols is obvious.

Let $n \geq 2$

$$D_2 O(2n, q): \quad v = b = \frac{q^{2n} - 1}{q - 1}; \qquad k = \frac{q^{2n-1} - 1}{q - 1}.$$

$$D_2 U(2n, q): \quad v = b = \frac{(q^n \sqrt{q} + 1)(q^n - 1)}{q - 1}; \quad k = q \frac{(q^{n-1} \sqrt{q} + 1)(q^{n-1} - 1)}{q - 1} + 1.$$

Let $n \geq 1$

$$D_2 H(2n+1, q): \quad v = b = \frac{(q^{n+1} - 1)(q^n + 1)}{q - 1}; \quad k = q \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1} + 1.$$

$$D_2 E(2n+1, q):^* \quad v = b = \frac{(q^{n+1} + 1)(q^n - 1)}{q - 1}; \quad k = q \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1} + 1.$$

*(Here assume $n \geq 2$)

$$D_2 S(2n+1, q): \quad v = b = \frac{q^{2n+2} - 1}{q - 1}; \qquad k = \frac{q^{2n+1} - 1}{q - 1}.$$

$$D_2 U(2n+1, q): \quad v = b = \frac{(q^n \sqrt{q} + 1)(q^{n+1} - 1)}{q - 1}; \quad k = q \frac{(q^{n-1} \sqrt{q} + 1)(q^n - 1)}{q - 1} + 1.$$

Clearly (see also [3], p. 28), it follows:

PROPOSITION 9.—$D_2 S(2n + 1, q)$ *is a symmetric 2-design.*

THEOREM 1.—*For* $n \geq 2$, $D_2 O(2n, q)$ *is a symmetric 2-design with the same parameters as* $D_2 S(2n - 1, q)$.

For the proof see [5]. However, an independent proof can be given noticing that the number of absolute hyperplanes through two points of a quadric $\mathcal{Q}_{2n}$ is a constant.

Remark: $D_2 O(2n, q)$ and $D_2 S(2n - 1, q)$ are NOT isomorphic. In fact, while all the "lines" (1) of $D_2 S(2n - 1, q)$ carry $q + 1$ points, lines of $D_2 O(2n, q)$ carry either $q + 1$ or 2 points. More precisely, in $D_2 O(2n, q)$ there are $\frac{(q^d - 1)(q^{d-2} - 1)}{(q - 1)(q^2 - 1)}$ $(q + 1)$-lines and $q^{d-1} \frac{(q^d - 1)}{2(q - 1)}$ 2-lines.

PROPOSITION 10.—*None of the remaining* $D_2(\pi)$*'s other than* $D_2 O(2n, q)$ *and* $D_2 S(2n + 1, q)$ *is ever a 2-design.*

The proof follows from some of the previous results.

METHOD N. 3

We can, however, find more 2-designs and a great many tactical configurations by modifying the above method slightly. Given a polarity π, take the absolute points as "points" and, as blocks, the set of points on every

(1) By definition, a "line" of a design is the intersection of all the blocks through two points.

absolute hyperplane P^{π} *excluding the point* P. The class of T.C.'s thus obtained will be denoted by $D_3(\pi)$. It is easy to prove that:

(*a*) $D_3 H(3, 3)$ is a 2–(16, 6, 2) symmetric design.

(*b*) $D_3 U(3, 4)$ is a 2–(45, 12, 3) symmetric design.

(*c*) None of the remaining $D_3(\pi)$'s is a 2–design.

The proof of (*b*) can be found in [2] and [4]. (*a*) is proved in a similar way. A computation on parameters proves (*c*).

Clearly, the complements of designs (*a*) and (*b*) are symmetric 2–designs, which could also be found independently (see for instance [4], p. 26).

OTHER POSSIBLE METHODS

4) For $0 \leq h < m \leq \frac{d-1}{2}$, take h–dimensional and m–dimensional totally isotropic subspaces as points and blocks, respectively (see [3], p. 291).

5) Let π be unitary: take non-absolute points as points and tangent lines as blocks.

6) Let π be symplectic or unitary: take non-absolute points as points and secants as blocks.

Methods 4), 5) and 6) lead to T.C.'s: $D_4(\pi)$, $D_5(\pi)$ and $D_6(\pi)$, respectively, that are never 2–designs.

For π unitary, another method which gives some 2–designs is illustrated in [4].

THEOREM 2.—*Let* $\Gamma(\pi)$ *be the group of polarity* π. *Then every T.C.* $D_i(\pi)$ $(i = 1, 2, 3, 4, 5, 6)$ *possesses a collineation group which contains an isomorphic copy of* $\Gamma(\pi)$.

Proof: The existence of a group endomorphism of $\Gamma(\pi)$ in the collineation group of each $D_i(\pi)$ is fairly obvious. The proof that this is in fact a monomorphism perhaps requires a little more effort.

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