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An intersection theorem in Banach spaces

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Analisi funzionale. — *An intersection theorem in Banach spaces.* Nota (*) di ALFONSO VIGNOLI (**), presentata dal Socio G. SANSONE.

SUMMARY. — Let the Banach space $X = A \oplus B$ be the direct sum of two subspaces $A, B \subset X$, and let $f: A \to X$, $g: B \to X$ be continuous mappings. A condition is given on f and g in order to ensure that the intersection $f(A) \cap g(B)$ is not empty.

1. INTRODUCTION. Let X be a real Banach space. Assume that $X = A \oplus B$ i.e. X is a direct sum of two subspaces $A \subset X$ and $B \subset X$.

Let $f: A \to X$ and $g: B \to X$ be continuous mappings. Granas in [1] states a sufficient condition in order that $f(A) \cap g(B) \neq \varnothing$.

The aim of the present paper is to extend Granas' result to a more general class of mappings.

We will use the following notations. Let $A \subset X$ be a bounded set of a metric space (X, d). By $\alpha(A)$ we denote the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε. (See Kuratowski [2]).

We will need the following two properties of the number α:

a) $\alpha(A) = 0$ iff A is precompact,

b) $\alpha(B + A) \leq \alpha(B) + \alpha(A)$, for any bounded subsets A, B of some Banach space X.

A continuous mapping $T: X \to X$ from a metric space (X, d) into itself is said to be *α–Lipschitz with constant* L, if for any bounded set $A \subset X$ we have

$$\alpha(T(A)) \leq L\,\alpha(A) \quad , \quad 0 \leq L < +\infty .$$

If the constant $L < 1$ the mapping T is said to be *α–contractive* (see Darbo [3]). In the case when $\alpha(T(A)) < \alpha(A)$, for any bounded set $A \subset X$, such that $\alpha(A) > 0$, the mapping T is called *densifying* (see [4]).

Clearly any completely continuous mapping (i.e. T is continuous and maps bounded sets into precompact ones) is α–Lipschitz with constant $L = 0$.

It is easy to verify that if T satisfies

$$d(T(x), T(y)) \leq L d(x, y) \quad , \quad \forall x, y \in X , \tag{1}$$

then T is also α–Lipschitz with constant L. If condition (1) holds for $L < 1$ then T is called *contractive*.

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Note that contractive and completely continuous mappings are densifying. Furthemore on a Banach space the sum of a contractive mapping (or more generally a densifying mapping) and a completely continuous mapping is densifying.

The following definition was introduced by Granas [5]. Let $T : X \to Y$ be a continuous mapping from a Banach space X into a Banach space Y. If the number

$$|T| = \limsup_{\|x\| \to \infty} \frac{\|T(x)\|}{\|x\|},$$

is finite then the mapping T is said to be *quasibounded* and $|T|$ is called the *quasinorm* of T. Evidently a continuous linear mapping $H : X \to Y$ is quasibounded and the quasinorm $|H|$ of H coincides with the norm $\|H\|$ of H.

The following theorem was proved in [6].

THEOREM A. *Let* $T : X \to X$ *be a quasibounded densifying mapping from a Banach space* X *into itself. Let* $|T| < 1$, *then the equation* $y = x + T(x)$ *has at least one solution for each* $y \in X$.

2. AN INTERSECTION THEOREM. Let $X = A \oplus B$ be a direct sum of two subspaces $A \subset X$ and $B \subset X$. By $P_A : X \to A$ we denote the projection mapping of X onto A and by $P_B : X \to B$ the projection mapping of X onto B. The mappings P_A, P_B are linear and we have

$$\|P_A(x)\| \leq \|P_A\| \|x\| \quad , \quad \|P_B(x)\| \leq \|P_B\| \|x\| , \qquad \forall x \in X,$$

where $\|P_A\|$, $\|P_B\|$ are the norms of P_A and P_B respectively.

THEOREM 1. *Let* $X = A \oplus B$ *and let* $f : A \to X$ *and* $g : B \to X$ *be such that*

$$f(a) = a + F(a), \quad \forall a \in A,$$

$$g(b) = b + G(b), \quad \forall b \in B,$$

where the mappings F *and* G *are* α*–Lipschitz with constants* L *and* L′ *respectively which satisfy*

$$L \|P_A\| + L' \|P_B\| < 1.$$

If the mappings F *and* G *are also quasibounded with quasinorms satisfying*

$$|F| \|P_A\| + |G| \|P_B\| < 1,$$

then $f(A) \cap g(B) \neq \emptyset$.

Proof. Since $X = A \oplus B$ any element $x \in X$ can be represented as follows

$$x = a - b \quad , \quad a \in A \quad , \quad b \in B.$$

Let $T = T_1 + T_2$, where $T_1 = F \circ P_A$ and $T_2 = - G \circ P_B$. The mapping $T : X \to X$ is densifying. Indeed, for any bounded subset $D \subset X$ such that

$\alpha(D) > 0$ we have (see property *b*) of Introduction)

$$\alpha(T(D)) \leq \alpha(F \circ P_A(D)) + \alpha(G \circ P_B(D)) \leq L\alpha(P_A(D)) + L'\alpha(P_B(D)) \leq$$
$$\leq L \| P_A \| \alpha(D) + L' \| P_B \| \alpha(D) < \alpha(D).$$

The mapping T is also quasibounded with quasinorm $|T| < 1$. The proof of this statement is essentially the same as that given by Granas in [1].

Hence by Theorem A there exists at least one element $x' \in X$ such that $x' + T(x') = 0$ i.e.

$$0 = a' - b' + F(a') - G(b'),$$

which implies $f(a') = g(b')$ and this proves the theorem.

The above theorem contains as a particular case the result by Granas [1] for F and G completely continuous. In this case the condition $L \| P_A \| + L' \| P_B \| < 1$ is trivially satisfied since $L = L' = 0$.

From Theorem 1 we obtain the following corollaries.

COROLLARY 1. *Let* $X = A \oplus B$ *and let* $f : A \to X$, $g : B \to X$ *be such that*

$$(2) \qquad \begin{aligned} f(a) &= a + F(a) + M(a), && \forall a \in A, \\ g(b) &= b + G(b) + N(b), && \forall b \in B, \end{aligned}$$

where the mappings M *and* N *are completely continuous and the mappings* F *and* G *are* α*–Lipschitz with constants* L *and* L′ *respectively such that*

$$L \| P_A \| + L' \| P_B \| < 1.$$

Let the mappings F *and* G *be also quasibounded with quasinorms satisfying*

$$|F| \| P_A \| + |G| \| P_B \| \leq \beta \quad , \quad 0 \leq \beta < 1,$$

and let the mappings M *and* N *be quasibounded with quasinorms such that*

$$|M| \| P_A \| + |N| \| P_B \| < 1 - \beta.$$

Then $f(A) \cap g(B) \neq \varnothing$.

Proof. Let $T = T_1 + T_2$, where $T_1 = F \circ P_A + M \circ P_A$, $T_2 = -G \circ P_B - N \circ P_B$. The mapping T is densifying. Indeed

$$\alpha(T(D)) \leq \alpha(F \circ P_A(D)) + \alpha(M \circ P_A(D)) + \alpha(G \circ P_B(D)) + \alpha(N \circ P_B(D)) \leq$$
$$\leq L\alpha(P_A(D)) + L'\alpha(P_B(D)) < \alpha(D).$$

The mapping T is also quasibounded and $|T| < 1$. Hence by Theorem 1 we get the result.

In Corollary 1 instead of (2) we could consider the following mappings

$$(2') \qquad \begin{aligned} f(\mu; a) &= a + F(a) + \mu M(a), \\ g(\mu; b) &= b + G(b) + \mu N(b), \end{aligned}$$

where the mappings F, G satisfy the same hypothesis of Corollary 1 and the quasinorms of M, N and the real number μ are such that

$$|\mu|(|M|\,\|P_A\| + |N|\,\|P_B\|) < 1 - \beta \quad , \quad 0 \leq \beta < 1 .$$

Then $f(\mu; A) \cap g(\mu; B) \neq \varnothing$. (Evidently for $\mu = 1$ we obtain Corollary 1). Let R denote the real numbers then we have

COROLLARY 2. *Let* $X = A \oplus B$ *and let* $f: R \times A \to X$ *and* $g: R \times B \to X$ *be such that*

$$f(\lambda; a) = a + \lambda F(a), \qquad \forall a \in A,$$

$$g(\lambda; b) = b + \lambda G(b), \qquad \forall b \in B,$$

where λ *is a real number such that* $|\lambda| \leq 1$ *and the mappings* F, G *are* α*–Lipschitz with constants* L, L′ *respectively, satisfying*

$$L\,\|P_A\| + L'\|P_B\| < 1 .$$

If the mappings F, G *are also quasibounded with quasinorms satisfying*

$$(3) \qquad |\lambda|(|F|\,\|P_A\| + |G|\,\|P_B\|) < 1,$$

then $f(\lambda; A) \cap g(\lambda; B) \neq \varnothing$.

COROLLARY 3. *Let* X, f, g *and* λ *be as in* Corollary 2. *If instead of the condition* (3) *the mappings* F, G *satisfy*

$$\|F(x)\| = o(\|x\|) \quad \text{as} \quad \|x\| \to \infty,$$

$$\|G(x)\| = o(\|x\|) \quad \text{as} \quad \|x\| \to \infty,$$

then $f(\lambda; A) \cap g(\lambda; B) \neq \varnothing$.

The condition $|\lambda| \leq 1$ in Corollary 2 and 3 is required in order that the mapping λT be densifying.

Corollary 3 in the particular case of $\lambda = 1$ and F, G completely continuous was proved by Granas [1].

While this paper was in print Petryshyn pointed out to me that similar results were obtained in [7] for P–compact operators.

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