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The Euler-Poincaré Characteristic and Pontrjagin Number of Einstein-Lorentzian Manifolds

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Geometria. — *The Euler–Poincaré Characteristic and Pontrjagin Number of Einstein–Lorentzian Manifolds.* Nota (*) di JOSEPH ZUND, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — In questa Nota, si considerano le proprietà topologiche di varietà Einstein–Lorentziane quadridimensionali che ammettono una metrica $g_{\mu\nu}$ definita dallo spazio–tempo della relatività generale. In modo particolare, si trattano le relazioni che esistono tra le caratteristiche Euler–Poincaré ed i numeri di Pontrjagin, quando le equazioni di Einstein sono assunte valide (*a*) per un campo gravitazionale « in vacuo » e (*b*) per altri campi fisici in presenza di materia od elettricità.

1. Let $\mathfrak{M}$ be a four dimensional differentiable manifold of class C^∞ which possesses a pseudo-Riemannian metric having p positive and $4-p$ negative squares when written in canonical form. Suppose $\mathfrak{M}$ admits a field of differentiable p–planes. A positive definite Riemannian metric $c_{\mu\nu}$ on $\mathfrak{M}$ induces on these p–planes a metric $a_{\mu\nu}$ and a metric $b_{\mu\nu}$ on the field of orthogonal $(4-p)$–planes. Hence $c_{\mu\nu} = a_{\mu\nu} + b_{\mu\nu}$, and by considering the tensor $l_{\mu\nu}(\lambda) \equiv c_{\mu\nu} + \lambda b_{\mu\nu}$, where λ is a real parameter, it can be shown that $l_{\mu\nu}$ defines a non-singular pseudo-Riemannian metric on $\mathfrak{M}$ for $\lambda \neq -1$. This metric is positive definite if $\lambda > -1$, and if $\lambda < -1$ it possesses p positive squares. For our hyperbolic normal Lorentzian metric we take $l_{\mu\nu}(-2) \equiv g_{\mu\nu}$. By employing these techniques Avez, [1, 2], derived the following integral formulae for the Euler-Poincaré characteristic $\chi(\mathfrak{M})$ and the Pontrjagin number $p[\mathfrak{M}]$ of a compact orientable pseudo-Riemannian manifold:

$$\chi(\mathfrak{M}) = \frac{(-1)^{[p/2]}}{32\,\pi^2} \int_{\mathfrak{M}} \Delta_E \cdot dv \tag{1}$$

$$p[\mathfrak{M}] = \frac{(-1)^{p}}{32\,\pi^2} \int_{\mathfrak{M}} \Delta_p \cdot dv\,, \tag{2}$$

where

$$\Delta_E \equiv \varepsilon_{\alpha\beta\gamma\delta}\,\varepsilon_{\mu\nu\varrho\sigma}\,R^{\alpha\beta\mu\nu}\,R^{\gamma\delta\varrho\sigma}$$

$$\Delta_p \equiv \frac{1}{4}\,R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta}{}_{\lambda\mu}\,\varepsilon^{\gamma\delta\lambda\mu}\,;$$

$\varepsilon_{\mu\nu\varrho\sigma}$ being the Levi-Civita dualizor and dv is the volume element (1). The Pontrjagin number is related to the Hirzebruch index $\tau(\mathfrak{M})$ by the formula

$$\tau(\mathfrak{M}) = \frac{1}{3}\,p\,[\mathfrak{M}]\,. \tag{3}$$

(*) Pervenuta all'Accademia il 19 agosto 1970.

(1) The Euler-Poincaré characteristic for $\mathfrak{M}$ was also given independently by Chern [4]; the Pontrjagin classes have been studied by Borel, [3], and Zund, [7, 8, 9]. Our notation closely follows that employed in [9].

2. We now restrict our considerations to a Lorentzian manifold, i.e. $g_{\mu\nu}$ has one positive and three negative squares in canonical form. The expressions for Δ_E and Δ_p may be simplified by re-writing them in terms of the right, left and double duals

$$R^{*}_{\mu\nu\varrho\sigma} \equiv \frac{1}{2} \varepsilon_{\varrho\sigma\alpha\beta} R_{\mu\nu}{}^{\alpha\beta}$$

$$^{*}R_{\mu\nu\varrho\sigma} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}{}_{\varrho\sigma}$$

$$^{*}R^{*}_{\mu\nu\varrho\sigma} = \frac{1}{4} \varepsilon_{\mu\nu\sigma\beta} \varepsilon_{\varrho\sigma\gamma\delta} R^{\alpha\beta\gamma\delta}.$$

Hence (1) and (2) become

$$\chi(\mathfrak{M}) = -\frac{1}{8\pi^2} \int_{\mathfrak{M}} R_{\mu\nu\varrho\sigma} {}^{*}R^{*\mu\nu\varrho\sigma} \, dv \tag{4}$$

$$p[\mathfrak{M}] = -\frac{1}{64\pi^2} \int_{\mathfrak{M}} R_{\mu\nu\varrho\sigma} R^{*\mu\nu\varrho\sigma} \, dv . \tag{5}$$

By employing the identity of Ruse and Lanczos

$$R_{\mu\nu\varrho\sigma} + {}^{*}R^{*}_{\mu\nu\varrho\sigma} = E_{\mu\varrho} g_{\nu\sigma} + E_{\nu\sigma} g_{\mu\varrho} - E_{\mu\sigma} g_{\nu\varrho} - E_{\nu\varrho} g_{\mu\sigma} ,$$

where

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{4} g_{\mu\nu} ,$$

the integrand of (4) reduces to

$$R_{\mu\nu\varrho\sigma} \cdot {}^{*}R^{*\mu\nu\varrho\sigma} = - R_{\mu\nu\varrho\sigma} R^{\mu\nu\varrho\sigma} + 4 R_{\mu\nu} R^{\mu\nu} - R^2.$$

Thus, if $R_{\mu\nu} = 0$, the Euler-Poincaré characteristic reduces to

$$\mathring{\chi}(\mathfrak{M}) = \frac{1}{8\pi^2} \int_{\mathfrak{M}} R_{\mu\nu\varrho\sigma} R^{\mu\nu\varrho\sigma} \, dv . \tag{6}$$

3. In previous papers, [7, 8, 9], we have considered the behavior of $\chi(\mathfrak{M})$, i.e. $\mathring{\chi}(\mathfrak{M})$ when Einstein's vacuum field equations are imposed on the Lorentzian manifold. We now consider the topological properties of Einstein-Lorentzian manifolds on which the Einstein equations for a physical (non-vacuum) scheme are imposed. These equations are

$$G_{\mu\nu} = \varkappa T_{\mu\nu} , \tag{7}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$$

is the Einstein tensor, $\varkappa$ is the cosmological constant, and $T_{\mu\nu}$ is the energy-momentum tensor corresponding to the prescribed physical scheme.

THEOREM A. *Let $\mathfrak{M}$ be a compact orientable four dimensional Einstein-Lorentzian manifold of class* C^∞ *on which the Einstein equations* (7) *are valid. Then*

$$\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M}) - \frac{\varkappa^2}{8\pi^2}\int_{\mathfrak{M}} \{4 T_{\mu\nu} T^{\mu\nu} - T^2\} \, dv \tag{8}$$

where $\overset{\circ}{\chi}(\mathfrak{M})$ *is defined by* (6) *and* $T \equiv g^{\mu\nu} T_{\mu\nu}$.

The proof is immediate by using the Ruse-Lanczos identity and noting that by virtue of the definition of $G_{\mu\nu}$ and (7) we have

$$R_{\mu\nu} R^{\mu\nu} = \varkappa^2 T_{\mu\nu} T^{\mu\nu}$$

and

$$R = -\varkappa T .$$

If $T_{\mu\nu} = 0$, then $\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M})$ and $\overset{\circ}{\chi}(\mathfrak{M})$ is precisely the Euler-Poincaré characteristic of $\mathfrak{M}$. If $T_{\mu\nu} \neq 0$ then we may regard the expression for $\overset{\circ}{\chi}(\mathfrak{M})$ in equation (6) as the purely geometric part of the Euler-Poincaré characteristic whenever $R_{\mu\nu\varrho\sigma}$ is non-expressible in terms of $R_{\mu\nu}$ (2). Henceforth we will explicitly make this assumption in our analysis.

Theorem A now yields three corollaries corresponding to the most common physical schemes considered in general relativity.

COROLLARY 1. *For the scheme of pure matter, e.g. dust,*

$$T_{\mu\nu} = \rho v_\mu v_\nu$$

where $\rho > 0$ *and* $v_\mu v^\mu = 1$.

If the hypotheses of Theorem A are satisfied, then

$$\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M}) - \frac{3\varkappa^2}{8\pi^2}\int_{\mathfrak{M}} \rho^2 \, dv .$$

COROLLARY 2. *For the scheme of a perfect fluid,*

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu - p g_{\mu\nu}$$

where ρ *and* $p > 0$, *and* $v_\mu v^\mu = 1$. *If the hypotheses of Theorem A are satisfied, then*

$$\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M}) - \frac{3\varkappa^2}{8\pi^2}\int_{\mathfrak{M}} (\rho + p)^2 \, dv .$$

COROLLARY 3. *For the scheme of an electromagnetic field*

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} (F_{\varrho\sigma} F^{\varrho\sigma}) - F_{\mu\tau} F^{\tau}_{\nu}$$

where $F_{\mu\nu}$ *is the electromagnetic bivector. If the hypotheses of Theorem A are satisfied, then*

$$\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M}) - \frac{\varkappa^2}{2\pi^2}\int_{\mathfrak{M}} |K|^2 \cdot dv$$

(2) A simple example of such an $R_{\mu\nu\varrho\sigma}$ occurs when $C_{\mu\nu\varrho\sigma} = 0$ and $R = 0$.

where

$$K \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} i F_{\mu\nu} \overset{*}{F}{}^{\mu\nu}.$$

In Corollary 3, two distinct cases arise according to whether $K = 0$ or $K \neq 0$. If $K = 0$ the electromagnetic field is said to be singular, and we have a radiation field. Hence in this case $\chi(\mathfrak{M}) = \overset{\circ}{\chi}(\mathfrak{M})$. If $K \neq 0$ the field is non-singular and the electromagnetic field is not a radiation field.

Each of the above corollaries follows directly by evaluating $4 T_{\mu\nu} T^{\mu\nu} - T^2$ for the indicated physical scheme. This computation is immediate in Corollaries 1 and 2, however in Corollary 3 it is useful to recall the involutory property of $T_{\mu\nu}$, i.e.

$$T_{\mu}{}^{\sigma} T_{\sigma\nu} = \frac{1}{4} | K |^2 g_{\mu\nu},$$

and the fact that $T \equiv g^{\mu\nu} T_{\mu\nu} = 0$ for arbitrary K.

The examples which we have considered imply that

$$\overset{\circ}{\chi}(\mathfrak{M}) \geq \chi(\mathfrak{M}) \tag{9}$$

with the equality occurring only for a singular electromagnetic field. Thus intuitively if we regard the Euler-Poincaré characteristic as a rough measure of the number of "holes" in $\mathfrak{M}$, it appears that the effect of a physical scheme would be to close up some of the " holes " in $\mathfrak{M}$. This interpretation may be made more precise by recalling Hopf's famous theorem, [6], that the Euler-Poincaré characteristic is the sum of the indices of singularity of a vector field of class C^∞ having only isolated singularities on $\mathfrak{M}$. If we denote the indices of singularity of a vacuum and non-vacuum Einstein-Lorentzian manifold $\mathfrak{M}$ by i_a and j_b respectively, then

$$\overset{\circ}{\chi}(\mathfrak{M}) = \sum_a i_a \quad , \quad \chi(\mathfrak{M}) = \sum_b j_b$$

and (9) becomes

$$\sum_a i_a \geq \sum_b j_b . \tag{10}$$

Suppose $\overset{\circ}{\chi}(\mathfrak{M}) = 0$. Then (9) requires that $\chi(\mathfrak{M}) \neq 0$ (unless the physical scheme is a singular electromagnetic field) and the vector field of class C^∞ on $\mathfrak{M}$ has singularities which are created by the physical scheme.

Suppose $\overset{\circ}{\chi}(\mathfrak{M}) \neq 0$. Then by the appropriate choice of a physical scheme, or by choosing particular values of the parameters in a given physical scheme, we can make

$$\overset{\circ}{\chi}(\mathfrak{M}) = \frac{\varkappa^2}{8\pi^2} \int_{\mathfrak{M}} \{4 T_{\mu\nu} T^{\mu\nu} - T^2\} \, dv . \tag{11}$$

Thus in this case $\chi(\mathfrak{M}) = 0$. On the other hand if we do not make the choice indicated in (11), then $\chi(\mathfrak{M}) = 0$, and the value of $\sum_b j_b$ will increase or decrease according to the sign of $\sum_a i_a$.

4. We now consider the effect of a physical scheme on the Pontrjagin number, and we denote by $\overset{\circ}{p}[\mathfrak{M}]$ the value of (6) subject to the condition $R_{\mu\nu} = 0$.

THEOREM B. *Let $\mathfrak{M}$ be a compact orientable four dimensional Einstein-Lorentzian manifold of class C^{∞} on which the Einstein equations* (7) *are valid. Then regardless of the physical scheme*

$$p[\mathfrak{M}] = \overset{\circ}{p}[\mathfrak{M}] \tag{12}$$

and hence

$$\tau[\mathfrak{M}] = \overset{\circ}{\tau}[\mathfrak{M}] . \tag{13}$$

The proof follows by noting that the integrand of (5), namely $R_{\mu\nu\varrho\sigma} R^{*\mu\nu\varrho\sigma}$, is independent of $R_{\mu\nu}$ by virtue of our assumption on the structure of $R_{\mu\nu\varrho\sigma}$. A direct verification of this fact can also be made by employing the spinor formalism of [9] to evaluate the integrand of (5). Equation (13) is just a restatement of (12) using the Hirzebruch-Thom formula (3). Hence the signature of the quadratic form generated by the cohomology cup product in $H^2(\mathfrak{M}; \mathbf{R})$ is unaffected by the introduction of a physical scheme in $\mathfrak{M}$.

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