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On the theory of fixed points for some classes of mappings III

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Analisi funzionale. — *On the theory of fixed points for some classes of mappings III.* Nota (*) di VASILE ISTRĂŢESCU e ANA ISTRĂŢESCU, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questo lavoro si studiano alcune condizioni sufficienti affinché un operatore abbia dei punti uniti; alcuni risultati generalizzano quelli di Rothe e Krasnoselskiĭ.

1. INTRODUCION. Let $\mathfrak{X}$ be a real or complex Banach space and C be a closed bounded set in $\mathfrak{X}$. If S is a completely continuous mapping of C into C, then according to the Shauder fixed point theorem S has a fixed point in C if C is convex. Using a variational principle, Schauder fixed point theorem was extended for the existence of solutions in certain balls B_r of the space $\mathfrak{X}$ for the nonlinear equation of the form

$$T(x) = S(x)$$

where S is completely continuous T is essentially a strongly monotone potential mapping. Another direction of extension of Schauder fixed point theorem was for the existence of fixed points for completely continuous mappings S of B_r into $\mathfrak{X}$ which on the boundary ∂B_r of B_r satisfy the condition

$$\| S(x) - x \|^2 \geq \| Sx \|^2 - \| x \|^2$$

These extensions are due by Rothe [6], Krasnoselskiĭ [5], Altman [1], [2], Kačurovski [4] and Petryschin [7].

The purpose of this Note is to obtain theorems of the above type for the class of densifying operators.

2. Let $\mathfrak{X}$ be a Banach space and E be a bounded set in $\mathfrak{X}$. We define [3] the Kuratovski number $\alpha(E)$ of E as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of E with balls with radius smaller than ε.

The properties of number $\alpha(\cdot)$ are the following:

1) $0 \leq \alpha(A) \leq$ diameter of A.

2) $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$.

3) $\alpha(A) = 0$, iff A is precompact.

(*) Pervenuta all'Accademia il 15 luglio 1970.

Definition 1. An operator T is called densifying if for each bounded set A

$$\alpha(\mathrm{TA}) < (\mathrm{A})$$

and is called α–contraction if there exists $k \in [0, 1)$ independent of A such that

$$\alpha(\mathrm{TA}) \leq k\alpha(\mathrm{A}) .$$

Remarks. It is clear that every completely continuous operator is α–contraction and that every α–contraction is densifying. These classes were considered by Darbo and Sadovski.

A more general class of operators was considered in [9] by

Definition 2. An operator is called locally power densifying, if for each bounded set $\mathrm{A} \subset \mathfrak{X}$ there exists an integer $n = n(\mathrm{A})$ such that

$$\alpha(\mathrm{T}^n \mathrm{A}) < \alpha(\mathrm{A})$$

and is called locally power contraction if for each bounded set A there exists $n = n(\mathrm{A})$ and $k \in [0, 1)$ such that $\alpha(\mathrm{T}^n \mathrm{A}) \leq k\alpha(\mathrm{A})$.

3. Generalizations of some fixed point theorems.

The definitions and notations of § 1–2 remain valid. An operator is called demicontinuous if $x_n \to x$ strongly in $\mathfrak{X}$ the $\mathrm{T}x_n \to \mathrm{T}x$ weakly in $\mathfrak{X}^*$ since we consider T to be an operator from $\mathfrak{X}$ to $\mathfrak{X}^*$.

THEOREM 3.1. *Let $\mathfrak{X}$ be a complex reflexive Banach space and $\mathrm{T}: \mathfrak{X} \to \mathfrak{X}^*$ be a mapping from $\mathfrak{X}$ to $\mathfrak{X}^*$ such that* T *is demicontinuous and*

$$|\langle \mathrm{T}x - \mathrm{T}y , x - y \rangle| \geq \beta \|x - y\|^2$$

for all $x, y \in \mathfrak{X}$ and some constant $\beta > 0$. Let S *be an α–contraction such that*

$$\frac{1}{\beta}(\mathrm{S}x - \mathrm{T}o) \in \{u , u \in \mathfrak{X}^*, \|u\| \leq r^*, r^* \leq r\} \tag{1}$$

for all $x \in \mathrm{B}_r$, $r > 0$ and $\alpha(\mathrm{SE}) < \beta\alpha(\mathrm{E})$.

Then there exists at least one point $x_0 \in \mathrm{B}_r$ such that

$$\mathrm{T}x_0 = \mathrm{S}x_0 .$$

Proof. The proof is similar to proof of theorem 1 in [7] and we give here only modifications in that proof for obtain our theorem.

From hypotheses and Zarantonelo-Browder theorem T^{-1} is well-defined on all $\mathfrak{X}^*$ and the operator $\mathrm{T}^{-1}\mathrm{S}$ maps B_r into B_r. Also

$$\|\mathrm{T}^{-1}\mathrm{S}y - \mathrm{T}^{-1}\mathrm{S}z\| \leq \frac{1}{\beta}(\|\mathrm{S}y - \mathrm{S}z\|) .$$

This gives that $T^{-1}S$ is a densifying operator on B_r and since it is continuous we find a fixed point x_0 such that

$$T^{-1}Sx_0 = x_0$$

and the theorem is proved.

Remark. The case when S is completely continuous, is theorem 1 of Petryshin [7]. Also theorem 2 of [7] has a variant in our context and we omit this.

Now, we wish to present some generalizations of classical theorem of Rothe [6], Krasnoselskiĭ [5] and Al'tman [1], [2]. These generalizations are obtained without use of the notion of the degree of a mapping in the sense of Leray-Schauder (see the proof of Petryshin of these theorems [7]).

THEOREM 3.2. *If* S *is a densifying continuous mapping form* B_r *to* $\mathfrak{X}$ *such that for every* $x \in B_r$

$$\|x - Sx\|^2 \geq \|Sx\|^2 - \|x\|^2$$

then S *has at least one fixed point in* B_r *for the case* $\mathfrak{X}$ *is a Hilbert space and* S *is* α*–contraction with* $k < \frac{1}{2}$ *for the case of Banach spaces.*

Proof. We define the retraction map of X on B_r

$$Ru = \begin{cases} u & \text{if } \|u\| \leq r \\ \dfrac{ru}{\|u\|} & \text{if } \|u\| \geq r \end{cases}$$

and consider the mapping $S_1(x) = RSx$ on B_r into B_r. By a the theorem of G. Darbo [3] p. 92, S_1 is a densifying operator and by a theorem of Sadovski [10] we find a fixed point x_0 in B_r, $S_1 x_0 = x_0$. Now, it is clear that x_0 is also a fixed point for S. Indeed since $x_0 \in B_r$, then either $\|x_0\| < r$ or $\|x_0\| = r$

$$x_0 = S_1 x_0 = RSx_0 = \begin{cases} Sx_0 & \text{if } \|Sx_0\| \leq r \\ r\dfrac{Sx_0}{\|Sx_0\|} & \text{if } \|Sx_0\| > r \end{cases}$$

Thus, it is sufficient to discuss the case when $\|Sx_0\| \geq r$, i.e. the equation

$$Sx_0 = \lambda_0 x_0 \qquad \lambda_0 = \frac{\|Sx_0\|}{r}.$$

If $\|x_0\| < r$ then we obtain clearly a contradiction. Thus remains only the case $\|x_0\| = r$ and thus $\lambda_0 \geq 1$. Clearly

$$\|x_0\|^2 - \|Sx_0\|^2 = \|x_0 - \lambda_0 x_0\|^2 = (1 - \lambda_0)^2 r$$

and

$$\|Sx_0\|^2 - \|x_0\|^2 = (\lambda_0 - 1) r.$$

Since $\lambda_0 \geq 1$, we must have $\lambda_0 = 1$ i.e. x_0 is a fixed point for S.

As special cases we obtain some generalizations of theorems proved by Rothe [6] and Krasnoselskiĭ [5].

THEOREM 3.3. *If* S *is a continuous densifying mapping from* B_r *to* $\mathfrak{X}$ *such that for every* $x \in \partial B_r$

$$\|Sx\| \leq \|x\|$$

then S *has at least one fixed point in* B_r *for* $\mathfrak{X}$ *a Hilbert space.*

THEOREM 3.4. *If* $\mathfrak{X}$ *is a Hilbert space and* S *be a continuous densifying mapping from* B_r *to* $\mathfrak{X}$ *such that for every* $x \in \partial B_r$

$$\langle Sx , x \rangle \leq \|x\|^2$$

thus S *has at least one fixed point in* B_r.

Remark. The essential role in the application of Darbo's theorem is the relation concerning the retraction R,

$$\| Ru - Ru' \| \leq \| u - u' \|$$

for the case of Hilbert spaces and

$$\| Ru - Ru' \| \leq 2 \| u - u' \|$$

for the case of Banach spaces. Perhaps, the relation $\| Ru - Ru' \| \leq \| u - u' \|$ is valid in general. But the authors are unable to prove this.

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