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Investigations in the topological method of Ważewski

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Equazioni differenziali. — *Investigations in the topological method of Ważewski.* Nota [(*)] di J. W. BEBERNES e J. D. SCHUUR [(**)], presentata dal Socio G. Sansone.

RIASSUNTO. — Il metodo di Ważewski s'è esteso all'equazione $y' = f(t, y)$ senza una condizione di unicità usando i teoremi di applicazioni di multivalori e di insiemi invarianti.

1. The topological method of Ważewski [1] is used to show the existence of at least one solution of $x' = f(t, x)$ which remains in an open set V on its maximal interval of existence. It is assumed that the equation satisfies hypotheses for the local existence and uniqueness of solutions on an open set U and that $V \subset U$. The method relies on the continuity of the consequent mapping, i.e. the mapping of V onto the boundary of V under the action of solutions.

If the assumption of uniqueness is dropped, then the consequent mapping may take a point onto a set. Hence, it becomes a set valued mapping. To preserve the method of Ważewski it is then necessary to show that the consequent mapping is upper semicontinuous, in the sense of set valued mappings, and also that the image of a point under this mapping is a compact and connected set.

Such a program has been carried out by Jackson and Klaasen [2] and it is also the purpose of this note. By using the theories of set valued mappings and of invariant sets we simplify their work. We also show that the upper semicontinuity of the consequent mapping depends upon the function appearing on the righthand side of the differential equation as well as upon the point P.

2. Let U be an open set in E^{n+1}. Points in U will be denoted by $P = (t_P, x_P)$, or $P_n = (t_n, x_n)$, or simply (t, x) where $t \in E^1$ and $x \in E^n$. On the set C (U) of all continuous functions mapping U into E^n with the compact open topology, i.e. uniform convergence on compact subsets, let $H = \{f_a : a \in A\}$ be an indexed subset. On $E^{n+1} \times C(U)$, use the product topology. For each $a \in A$ we have

$$x' = f_a(t, x). \tag{I_a}$$

Let $\varphi^a(t, P) = \varphi^a(t; t_P, x_P)$, or simply $\varphi(t)$, denote a solution of (I_a) with $\varphi^a(t_P, P) = x_P$.

The index a will be omitted when a single equation is considered.

Let ∂U denote the frontier of U. Let V be an open subset of U, let ∂V be the frontier of V in E^{n+1} and F (V) be the frontier of V relative to U.

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Let $\bar{V} = V \cup F(V)$, let $U - V$ be the complement of V in U, and let $\rho(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ where $|\cdot|$ is a norm in E^{n+1}.

A point $Q \in F(V)$ is a *consequent* of a point $P \in V$ (relative to (1)) if there exists a solution $\varphi(t, P)$ of (1) and b with $t_P < b \le t_Q$ such that $\varphi(t, P)$ is defined on $[t_P, t_Q]$, $(t, \varphi(t, P)) \in V$ for $t_P < t < b$, $(t, \varphi(t, P)) \in F(V)$ for $b \le t \le t_Q$, and $(t_Q, \varphi(t_Q, P)) = Q$. The point $Q \in F(V)$ is a *consequent* of a point $P \in F(V)$ (and P may equal Q) if there exists a solution $\varphi(t, P)$ of (1) such that $(t, \varphi(t, P)) \in F(V)$ for $t \in [t_P, t_Q]$ and $\varphi(t_Q, P) = x_Q$.

Consequents will also be called *points of egress* from V. An egress point Q is a *strict egress point* if, for every solution $\varphi(t, Q)$, $c_\varphi = \sup\{t : (s, \varphi(s, Q)) \in F(V), t_Q \le s \le t\} < \infty$ and there exists a sequence $\{t_n\}$ such that $t_n \to c_\varphi$, $t_n > c_\varphi$ and $(t_n, \varphi(t_n, Q)) \in U - \bar{V}$. For $(P, f) \in \bar{V} \times H$, let $G(P, f)$ be the set of consequents of P relative to (1). Then $G : \bar{V} \times H \to F(V)$ is the *consequent mapping*. In our theorems, $G(P, f)$ will be nonempty. These definitions follow Jackson and Klaasen.

A set valued mapping G from X into Y, X and Y metric spaces, is *upper semicontinuous* (USC) at $x \in X$ if (i) $G(x)$ is compact and (ii) for any open set $S \supset G(x)$ there exists a neighborhood T of x such that $G(T) \subset S$. If G is USC at each $x \in X$, then G is USC on X.

3. We shall show that the consequent mapping is USC.

THEOREM 1. *Let* G *be the consequent mapping from* $\bar{V} \times H$ *into* $F(V)$ *and let* $(P_0, f_0) \in \bar{V} \times H$. *Assume that all points of* $G(P_0, f_0)$ *are points of strict egress and assume that no solution of* (I_0) *which passes through* P_0 *approaches* $\partial V - F(V)$ *for* $t > t_0$. *Then* G *is* USC *at* (P_0, f_0).

Proof. Let $\varphi(t)$ be a solution of (I_0) with maximal interval of existence (α, β). As $t \to \beta$, $(t, \varphi(t)) \to \partial U$. If, additionally, $(t, \varphi(t)) \subset V$ for $t_0 \le t < \beta$, then $(t, \varphi(t)) \to \partial V - F(V)$ as $t \to \beta$. Thus, our hypotheses imply that $(t, \varphi(t))$ intersects $F(V)$ at some $t > t_0$ and that $G(P_0, f_0)$ is nonempty.

$G(P_0, f_0)$ is compact. Let $\{Q_n\}$ be a sequence of points in $G(P_0, f_0)$ and let $\varphi_n(t) = \varphi_n(t, P_0)$ be a solution of (I_0) such that $\varphi_n(t_n) = Q_n$. By the Kamke convergence theorem [3, p. 14] there is a subsequence $\{\varphi_k(t)\}$ which converges to a solution $\varphi_0(t) = \varphi_0(t, P_0)$ of (I_0). Since $\varphi_0(t)$ intersects $F(V)$ and every point of $G(P_0, f_0)$ is a strict egress point there exists a $c > t_0$ such that $\varphi_0(t)$ is defined on $[t_0, c]$ and $(c, \varphi_0(c)) \in U - V$. Thus, for k sufficiently large, $(c, \varphi_k(c)) \in U - \bar{V}$ which implies $t_0 \le t_k < c$. Choose now a subsequence $\{t_m\}$ of $\{t_k\}$, and the corresponding subsequence $\{\varphi_m(t)\}$ of $\{\varphi_k(t)\}$, such that $t_m \to b \le c$. Then $\varphi_m(t_m) \to \varphi_0(b)$. Let $Q_0 = (b, \varphi_0(b))$, then $Q_m \to Q_0$ and $Q_0 \in G(P_0, f_0)$.

To show that G satisfies (ii) in the definition of USC we can construct a proof by contradiction, similar to the one above, again based on the Kamke convergence theorem.

Next we examine when the image of a point under the consequent mapping is connected. This is similar to the Kneser theorem on the connectedness of cross sections of solution funnels, and in fact we state Kneser's theorem as a corollary to theorem 3. To prove our theorem we use a technique of Yorke [4].

A set $S \subset V$ is *positively weakly invariant* if, through each point $P \in S$ there exists at least one solution $\varphi(t, P)$ of (1) whose positive semitrajectory (i.e., $\{(t, \varphi(t, P)) : t \geq t_p$ and t lies in the maximal interval of existence of $\varphi(t, P)\}$) lies in S. *Negatively weakly invariant* is defined in a similar manner and a set which is both positively and negatively weakly invariant is *weakly invariant*. We note that S is weakly invariant if and only if S is a union of trajectories.

THEOREM 2 [4, p. 359]. *If* R *and* S *are positively weakly invariant and closed relative to* V *and if* $V = R \cup S$, *then* $R \cap S$ *is positively weakly invariant.*

THEOREM 3. *Let* G *be the consequent mapping from* $\bar{V} \times H$ *into* $F(V)$ *and let* $(P, f) \in \bar{V} \times H$. *If every solution of* (1) *which passes through* P *intersects* $F(V)$ *at some* $t \geq t_p$ *and all points of* $G(P, f)$ *are points of strict egress, then* $G(P, f)$ *is connected.*

Proof. If $P \in F(V)$ the proof is immediate. So suppose that $P \in V$ and suppose that $G(P, f)$ is not connected. Using Theorem 1 we may assume that $G(P, f) = C_1 \cup C_2$ where C_1 and C_2 are two disjoint nonempty sets which are compact relative to U.

Let $\Phi(Q)$ denote a trajectory of a solution of (1) through Q. Let $R = \{\Phi(Q) \subset V : \rho(\Phi(Q), C_1) \leq \rho(\Phi(Q), C_2)\}$ and let S be a similar set with the inequalities reversed. Then R and S are positively weakly invariant, closed relative to V, and $V = R \cup S$. By Theorem 2, $R \cap S$ is positively weakly invariant.

Since trajectories containing P intersect both C_1 and C_2, $P \in R \cap S$. Hence, there exists $\varphi(t, P)$ such that $(t, \varphi(t, P)) \in R \cap S$ on $[t_p, \delta)$, the right maximal interval on which $(t, \varphi(t, P)) \in V$. Let $[t_p, \beta)$ be the right maximal interval of existence of $\varphi(t, P)$ (with respect to U). Since $\varphi(t, P)$ leaves V, $\delta < \beta$, and $(t, \varphi(t, P))$ intersects $F(V)$ at, say, C_1. Then $0 = \rho(\Phi(P), C_1) = \rho(\Phi(P), C_2) > 0$. From this contradiction we conclude that $G(P, f)$ is connected.

COROLLARY (Kneser [3, p. 15]). *Let* $(P, f) \in U \times H$. *If every solution of* (1) *passing through* P *exists for* $t_p \leq t \leq t_1$, *then* $F(t_1, P) = \{(t_1, \varphi(t_1)) : \varphi(t)$ *is a solution of* (1) *passing through* $P\}$ *is compact and connected.*

4. The method of Ważewski is phrased in terms of retraction mappings, but it is often applied in the following form.

THEOREM 4. *Let* $A \subset V$ *be a connected and compact subset relative to* U *and let* $J \subset H$ *be compact and connected. If, for each* $(P, f) \in A \times J$, *all points of* $G(P, f)$ *are points of strict egress from* V *and if* $G(A, J)$ *is not connected, then there exists a* $(P_0, f_0) \in A \times J$ *and a solution* $\varphi(t, P_0)$ *of* (I_0) *such that* $(t, \varphi(t, P_0)) \in V$ *for all t in the right maximal interval of existence of* $\varphi(t, P_0)$.

Proof. If, for each $(P, f) \in A \times J$, every solution of (1) which passes through P intersects F (V) at some $t > t_p$, then we may use Theorems 1 and 3 and the fact that a USC mapping preserves compactness and connectedness (see for example [5]) to conclude that G (A, J) is compact and connected.

The language of set valued mappings also simplifies Jackson's and Klaasen's Ważewski-type theorem [2, Theorem 1]. Let A and B be subsets of X with $B \subset A$. If there exists a USC mapping G of A into B such that $x \in G(x)$ for each $x \in B$, then B is a *set valued retract* of A and G is a *set valued retraction* from A into B.

THEOREM 5. *Let* S *denote the set of egress points of* V *relative to* (1) *and assume that all egress points are strict. If there exists a* $Z \subset V \cup S$ *such that* $Z \cap S$ *is a set valued retract of* S *but not of* Z, *then there is a point* $P \in Z$ *and a solution* $\varphi(t, P)$ *of* (1) *such that* $(t, \varphi(t, P)) \in V$ *for all t in the right interval of existence of* $\varphi(t, P)$.

Proof. Assume that for all $P \in Z$ and every solution $\varphi(t, P)$ of (1), $\varphi(t, P)$ intersects F (V) at some $t > t_p$. Then the consequent mapping G is a USC mapping from Z into S. Let H be the set valued retraction from S into $Z \cap S$ given by the hypotheses. Then $HG : Z \to Z \cap S$ in USC (see [5]) and for each $P \in Z \cap S$, $P \in HG(P)$ since $P \in H(P) \subset HG(P)$. Thus $Z \cap S$ is a retract of Z which is a contradiction.

References

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