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# Rendiconti

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## Autonomous System of Two Ordinary Differential Equations With Homogeneous Right-Hand Sides

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**Equazioni differenziali.** — *Autonomous System of Two Ordinary Differential Equations With Homogeneous Right-Hand Sides.* Nota (*) di RUDOLF KURTH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Le soluzioni di una certa classe di equazioni differenziali sono caratterizzate qualitativamente e le formule sono date per le loro computazioni numeriche.

## § 1. THE PROBLEM

Let $R^2$ be the cartesian plane, and $f: R^2 - \{0\} \to R^2$ be a given vector function which, *first*, has continuous derivatives with respect to the components of the independent variable and, *secondly*, is homogeneous of degree $\alpha$; *thirdly*, it is supposed that the vectors $x \in R^2 - \{0\}$ and $f(x)$ are never collinear. Under these hypotheses the solutions of the differential equation

$$dx/dt = f(x)$$

(with $x \in R^2 - \{0\}$) will qualitatively be characterized, and explicit expressions will be given for their numerical computation.

The case that there are points $x$ in $R^2 - \{0\}$ at which the vectors $x$ and $f(x)$ are collinear has extensively been discussed in the literature. (See the references). The case considered in the present note has apparently attracted less attention.

## § 2. STANDING NOTATION

Let $r, \vartheta$ be polar coordinates in the punctured plane, $R^2 - \{0\}$,

$$\left.\begin{aligned} r &= |x|, \\ u &= x/r, \\ u' &= du/d\vartheta, \end{aligned}\right\} \qquad \text{(Def.)}$$

$$\left.\begin{aligned} \varphi(\vartheta) &= u' \cdot f(u), \\ \psi(\vartheta) &= u \cdot f(u) \end{aligned}\right\} \qquad \text{(Def.)}$$

(where the dot denotes the scalar multiplication of two vectors), and define a new " time variable ", $s$, by

$$\begin{cases} ds/dt = r^{\alpha-1}, \\ s = 0 \qquad \text{when} \quad t = 0. \end{cases}$$

(*) Pervenuta all'Accademia il 15 luglio 1970.

With these definitions, the system of differential equations of § 1 takes the form

$$\begin{cases} \mathrm{d}r/\mathrm{d}s = r\cdot\psi(\vartheta), \\ \mathrm{d}\vartheta/\mathrm{d}s = \varphi(\vartheta) \end{cases}$$

where $\varphi$ and $\psi$ are continuously differentiable periodic functions with periods $2\pi$. The following discussion will always refer to both these equations.

## § 3. QUALITATIVE RESULTS

The qualitative properties of the solutions of equations are summarized in the following:

THEOREM. (i) *For any given initial values* $(\vartheta_0, r_0)$ *(with* $r_0 > 0$*), the system has a unique solution which is defined for all values of* $s$.

(ii) Let

$$\left.\begin{aligned} A &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\vartheta}{\varphi(\vartheta)}, \\ B &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\psi(\vartheta)}{\varphi(\vartheta)}\,\mathrm{d}\vartheta, \end{aligned}\right\} \qquad \text{(Def.)}$$

$$\tau = s/A, \qquad \text{(Def.)}$$

C denote a certain positive real, and $\chi_1, \chi_2, \chi_3, \chi_4$ be certain periodic real function, with periods $2\pi$, which are defined and have continuous derivatives on the whole real axis. Then,

$$\begin{cases} \vartheta - \vartheta_0 = \tau + \chi_1(\tau), \\ \log(r/r_0) = B\tau + \chi_2(\tau), \end{cases}$$

$$t = \begin{cases} A r_0^{-(\alpha-1)}\cdot\{\chi_3(0) - e^{-(\alpha-1)B\tau}\chi_3(\tau)\} & \text{if } (\alpha-1)B \neq 0, \\ A r_0^{-(\alpha-1)}\cdot\{C\tau + \chi_4(\tau)\} & \text{if } \alpha - 1 \neq 0, \quad B = 0, \\ s = A\tau & \text{if } \alpha - 1 = 0. \end{cases}$$

Thus, if $(\alpha - 1)B \neq 0$, the $\tau$–axis is mapped only onto a proper sub-interval of the $t$–axis.

(iii) Let $k$ be any natural number, and $\vartheta_1$ and $\vartheta_2 (> \vartheta_1)$ be any two reals: then the arcs

$$\{(\vartheta, r) \mid \vartheta_1 \leq \vartheta \leq \vartheta_2\} \qquad \text{and} \qquad \{(\vartheta, r) \mid \vartheta_1 + 2\pi k \leq \vartheta \leq \vartheta_2 + 2\pi k\}$$

are geometrically similar, the factor of similarity being $e^{2\pi k B}$.

*Proof.* Choose the system of polar coordinates $(\vartheta, r)$ in such a way that $\vartheta_0 = 0$, and let

$$\Phi(\vartheta) = \int_0^{\vartheta} \frac{d\vartheta}{\varphi(\vartheta)} \cdot \qquad \text{(Def.)}$$

Then, by § 2,

$$\Phi(\vartheta) = s$$

for all real $s$. The mapping $\Phi$ of the real line into itself is one-to-one and onto. Hence the first half of Assertion (i).

With $\varphi$, also $1/\varphi$ is periodic with period $2\pi$. Therefore,

$$\Phi(\vartheta) = A\vartheta + A \cdot \tilde{\Phi}(\vartheta) \qquad \text{(Def.)}$$

where $\tilde{\Phi}$ is a periodic function of period $2\pi$.

Introduce new variables, $\tau$ and $\eta$, by

$$\left.\begin{aligned} \tau &= s/A, \\ \eta &= \vartheta - \tau. \end{aligned}\right\} \qquad \text{(Def.)}$$

Then,

$$\vartheta + \tilde{\Phi}(\vartheta) = \tau$$

and

$$\eta = \tilde{\Phi}(\tau + \eta).$$

Since $\eta = \chi(\tau)$ is uniquely determined by $\tau$, the last equation implies that $\chi$ is periodic with period $2\pi$, and therefore

$$\vartheta = \tau + \chi(\tau),$$

which is the first part of Assertion (ii) (with $\chi_1 = \chi$).

By § 2,

$$\frac{dr}{d\vartheta} = r \frac{\psi(\vartheta)}{\varphi(\vartheta)}:$$

hence,

$$r = r_0 \cdot e^{\Psi(\vartheta)}$$

where

$$\Psi(\vartheta) = \int_0^{\vartheta} \frac{\psi(\vartheta)}{\varphi(\vartheta)} d\vartheta. \qquad \text{(Def.)}$$

Since $\psi/\varphi$ is periodic with period $2\pi$,

$$\Psi(\vartheta) = B\vartheta + \tilde{\Psi}(\vartheta) \qquad \text{(Def.)}$$

where $\tilde{\Psi}$ is a periodic function of period $2\pi$.

Consequently,

$$r = r_0 \cdot e^{B\vartheta} \cdot e^{\tilde{\Psi}(\vartheta)},$$

which implies Assertion (iii) and, since

$$\vartheta = \tau + \chi(\tau),$$

the second parts of Assertions (i) and (ii), with

$$\chi_2(\tau) = B \cdot \chi(\tau) + \tilde{\Psi}(\tau + \chi(\tau)).$$

The last part of Assertion (ii) follows from

$$t = \int_0^s r^{1-\alpha}\, ds$$

$$= A r_0^{1-\alpha} \cdot \int_0^\tau e^{-(\alpha-1)B\tau - (\alpha-1)\chi_2(\tau)}\, d\tau :$$

If $\alpha \neq 1$, expand the continuously differentiable periodic function $e^{-(\alpha-1)\chi_2(\tau)}$ in a Fourier series, which converges uniformly towards this function, and substitute the series in the right-hand side of the last equation: an integration term by term yields the last part of Assertion (ii).

## § 4. FORMULAE FOR COMPUTING THE SOLUTIONS

Since the periodic functions $\chi_1, \chi_2, \chi_3, \chi_4$ have continuous derivatives, their Fourier series converge towards the generating functions:

$$\left.\begin{aligned} \chi_k(\tau) &= \sum_{n=-\infty}^{+\infty} a_n^{(k)} e^{in\tau} \\ a_n^{(k)} &= \frac{1}{2\pi} \int_0^{2\pi} \chi_k(\tau)\, e^{-in\tau}\, d\tau, \end{aligned}\right\} \qquad \begin{aligned} k &= 1, 2, 3, 4, \\ i &= \sqrt{-1}. \end{aligned}$$

The Fourier coefficients, $a_n^{(k)}$, become accessible to numerical computation by introducing $\vartheta$ as a new integration variable in the above expressions. By § 3,

$$\left\{\begin{aligned} &\chi(\vartheta) = \chi_1(\tau) = -\tilde{\Phi}(\vartheta), \\ &\tau = \Phi(\vartheta)/A, \\ &\tau + \chi(\tau) = \vartheta, \\ &d\tau = ds/A = \frac{1}{A} \frac{d\vartheta}{\varphi(\vartheta)}. \end{aligned}\right.$$

Hence,

$$a_n^{(1)} = -\frac{1}{2\pi A}\int_0^{2\pi} \frac{\tilde{\Phi}(\vartheta)}{\varphi(\vartheta)}\, e^{-in\Phi(\vartheta)/A}\, d\vartheta ,$$

$$a_n^{(2)} = \frac{1}{2\pi A}\int_0^{2\pi} \frac{\tilde{\Psi}(\vartheta)}{\varphi(\vartheta)}\, e^{-in\Phi(\vartheta)/A}\, d\vartheta ,$$

$$a_n^{(3)} = \frac{1}{2\pi A}\,[(\alpha - 1)\,B - in]^{-1} \cdot$$

$$\cdot \int_0^{2\pi} \exp\{(\alpha - 1)\,B\tilde{\Phi}(\vartheta) - (\alpha - 1)\,\tilde{\Psi}(\vartheta) - in\,\Phi(\vartheta)/A\}\,\frac{d\vartheta}{\varphi(\vartheta)}$$

$$\begin{cases} a_0^{(4)} = 0 , \\ a_n^{(4)} = \dfrac{1}{2\pi A}\,\dfrac{i}{n}\displaystyle\int_0^{2\pi} \exp\{-(\alpha - 1)\,\tilde{\Psi}(\vartheta) - in\,\Phi(\vartheta)/A\}\,\frac{d\vartheta}{\varphi(\vartheta)} \quad \text{if} \quad n \neq 0 . \end{cases}$$

The number C mentioned in the theorem of § 3 is

$$C = \frac{1}{2\pi A}\int_0^{2\pi} \exp\{-(\alpha - 1)\,\tilde{\Psi}(\vartheta)\}\,\frac{d\vartheta}{\varphi(\vartheta)} .$$

## References


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