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Fixed points of noncontinuous mappings

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Analisi matematica. — *Fixed points of noncontinuous mappings* (*). Nota (**) di ARRIGO CELLINA, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si danno teoremi di punto fisso per applicazioni non necessariamente continue.

Purpose of the present paper is to prove two fixed point theorems for mappings that are not necessarily demi-continuous (and therefore not necessarily continuous).

§ 1. In this section X is a Hausdorff locally convex space, X^* its dual and $\langle \cdot , \cdot \rangle : X^* \times X \to R$ is the pairing between X and X^*. For $M \subset X$, CK (M) is the set of non-empty compact and convex subsets of M. We also set δM to be the algebraic boundary of $M \subset X$ [2].

A [multivalued] mapping Γ is called [upper] demi-continuous [1], [2] if for every x and every open half space H in X containing $\Gamma(x)$, there exists a neighborhood N of x such that $\Gamma(\xi) \subset H$ for each $\xi \in N$.

A [multivalued] mapping Γ is called partially closed [4] if from $x_\delta \to x$, $y_\delta \in \Gamma(x_\delta)$, $y_\delta \to y$, $\delta \in \Delta$, it follows that $L(x, y) \cap \Gamma(x) \neq \varnothing$, where $L(x, y) = \{x + \lambda(y - x) : \lambda \geq 0\}$, and x_δ and y_δ are generalized sequences.

It is not difficult to see that if Γ is upper demi-continuous, it is partially closed.

The following results are known:

THEOREM 1 (Glebov [4]). *Let* K *be a compact and convex subset of* X, $\Gamma : K \to CK(K)$ *be partially closed. Then* Γ *has a fixed point in* K.

THEOREM 2 (Ky Fan [3]). *Let* K *be a compact and convex subset of* X, $\Gamma : K \to CK(X)$ *be upper demi-continuous. Assume that for every* $x \in \delta K$, *there exists a* $y \in \Gamma(x)$ *such that* $L(x, y)$ *contains a point of* K *distinct from* x. *Then* Γ *has a fixed point in* K.

The fact that a theorem holds, that contains both theorem 1 and 2, seems natural. For this purpose we consider mappings Γ that are continuous in the following sense:

(D_1) for each x in the domain of Γ and each continuous linear functional g that strictly separates x and $\Gamma(x)$, there exists a neighborhood U of O such that g strictly separates $x + U$ and $\Gamma(x + U)$.

We have the following

PROPOSITION 1. *i) if* (D_1) *holds,* Γ *is partially closed. ii) if* Γ *is partially closed and its range is contained in a compact set,* (D_1) *holds.*

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(**) Pervenuta all'Accademia il 24 luglio 1970.

Proof. i): let $x_\delta \to x$, $y_\delta \in \Gamma(x_\delta)$, $y_\delta \to y$, $\delta \in \Delta$; we claim $L(x, y) \cap \Gamma(x) = \varnothing$. Suppose this is not true. Then by a basic separation theorem, there exists a $g \in X^*$, a real $\varepsilon > 0$, such that $\inf \langle g, \Gamma(x)\rangle \geq k > k - \varepsilon \geq \sup \langle g, L(x, y)\rangle$. It follows that $\langle g, y\rangle \leq \langle g, x\rangle$. By ($D_1$) there exists a $U : \inf \langle g, \Gamma(x+U)\rangle \geq k_1 > k_1 - \varepsilon_1 \geq \sup \langle g, x+U\rangle$ for some real k_1 and $\varepsilon_1 > 0$. But then there exist δ' and $\delta'' \in \Delta$, such that $\delta' \prec \delta$ implies $x_\delta \in U$ and $\delta'' \prec \delta$ implies $y_\delta \in y + U$ so that for some δ''', $\delta' \prec \delta'''$, $\delta'' \prec \delta'''$, we have $y_{\delta'''} \in \Gamma(x+U) \cap y + U$, that contradicts the above inequality.

ii): assume that (D_1) is not true. Let U_δ, $\delta \in \Delta$, be the neighborhood filterbase at O. There exist $g \in X^*$, $\xi_\delta \in U_\delta$, $y_\delta \in \Gamma(\xi_\delta)$, a real k and $\varepsilon > 0$, such that $\inf \langle g, \Gamma(x)\rangle \geq k > k - \varepsilon \geq \langle g, x\rangle$, while $\langle g, \xi_\delta\rangle \geq \langle g, y_\delta\rangle - \varepsilon_\delta$, where $\varepsilon_\delta \downarrow 0$. We can assume that $y_\delta \to y$: it follows then that $\langle g, x\rangle \geq \langle g, y\rangle$. By assumption

$$\inf \langle g, \Gamma(x)\rangle \leq \sup \langle g, L(x, y)\rangle = \langle g, x\rangle + \sup_{\lambda \geq 0} \lambda \langle g, y - x\rangle = \langle g, x\rangle$$

contradicting the hypothesis on g.

THEOREM 3. *Let $K \subset X$ be compact and convex and $\Gamma : K \to CK(X)$ be such that (D_1) holds. Assume that for every $x \in \delta K$, there exists a $y_x \in \Gamma(x)$ such that $L(x, y_x)$ contains a point of K distinct from x.*

Then Γ has a fixed point in K.

Proof. The proof follows a pattern developed by Browder [2]. Assume that for every $x \in K$, $x \notin \Gamma(x)$. Then to each x we can associate a neighborhood of x, V_x, and a $g_x \in X^*$: $\sup \langle g_x, V_x\rangle < \inf \langle g_x, \Gamma(V_x)\rangle$. Let $\{V_{x_i}\}_{i=1}^n$ be a finite subcovering of the open covering $\{V_x\}$ and let $\{p_i(\cdot)\}$ be a partition of unity subordinated to $\{V_{x_i}\}$. Consider the continuous map $f : X \to X^*$ defined by

$$f(x) = \sum_{i=1}^{n} p_i(x) \cdot g(x_i).$$

By a result in [2], there exists a $x^0 \in K$:

$$\langle f(x^0), x - x^0\rangle \leq 0 \quad , \quad \forall x \in K. \tag{1}$$

Consider $f(x^0)$: it is a convex combination of those $g(x_i)$ such that $x^0 \in V_{x_i}$. By the definition of V_x then, $\langle g(x_i), x^0\rangle < \inf \langle g(x_i), \Gamma(x^0)\rangle$ so that $\langle f(x^0), x^0\rangle < \inf \langle f(x^0), \Gamma(x^0)\rangle$ or

$$\langle f(x^0), x^0 - \Gamma(x^0)\rangle \leq 0$$

i.e. $\langle g(x^0), L(x^0, y_{x^0})\rangle \leq 0$.

The last inequality and the assumption on $L(x, y_x)$ contradict (1).

Using Proposition 1 it is not difficult to see that Theorem 3 contains both Theorems 1 and 2.

§ 2. In this section we assume that X is a normed linear space and we restrict our attention to single-valued functions. We shall use the notations $B[x, \varepsilon]$ and $B[A, \varepsilon]$ to mean an open ε–ball about x and an ε–neighborhood of $A \subset X$ respectively. Set

$$J(x) = \{y \in X^* : \langle y , x\rangle = |x|^2 ; |y| = |x|\}.$$

We consider mappings $f : K \to X$ with the following property:

(D_2) For each $x \in K$, $\exists j_x \in J(f(x) - x)$ such that the real valued function $g_x : K \to R$ defined by

$$g_x(\xi) = \langle j_x , f(\xi)\rangle$$

is lower semi-continuous at $\xi = x$.

It is easy to see that there are no implications between (D_1) and (D_2). Moreover while a demicontinuous function satisfies both (D_1) and (D_2), the union of (D_1) and (D_2) is not equivalent to demicontinuity.

We have the following theorem

THEOREM 4. *Let* $K \subset X$ *be compact and convex,* $f : K \to X$ *be such that* (D_2) *holds; moreover assume that for each* $x \in K$, $\langle j_x , f(x)\rangle \leq \sup \langle j_x , K\rangle$. *Then* f *has a fixed point in* K.

Proof. Assume that $\forall x \in K , f(x) \neq x$. For each x there exists a $\delta(x) > 0$ (we assume $\delta(x) < 1/3 \, |f(x) - x|$) such that $\langle j_x , f(\xi) - x\rangle > 2/3 \, |f(x) - x|^2$ for all $\xi \in B[x , \delta(x)]$. Let us consider the open covering $\{B[x , \delta(x)/2]\}$ and let $\{B[x_i , \delta_i/2]\}_{i=1}^n$ be a finite subcovering (where $\delta_i = \delta(x_i)$). Let $\{p_i\}_{i=1}^n$ be a partition of unity subordinated to this last covering. For each x_i let $y_i \in K$ be such that $\langle j_{x_i} , y_i\rangle = \langle j_{x_i} , f(x_i)\rangle$. Consider the continuous map

$$g(x) = \sum_{i=1}^{n} p_i(x) \, y_i .$$

Since $g : K \to K$, there exists in K a $x^0 = g(x^0)$.

Let $k = 1 , 2 , \cdots , s$ be such that $p_{ik}(x^0) \neq 0$, so that $x^0 \in B[x_{ik} , \delta_{ik}/2]$. Let $\delta_{i^0} = \max\{\delta_{ik} : k = 1 , \cdots , s\}$. Then

$$d(x_{ik} , x_{i^0}) \leq d(x_{ik} , x^0) + d(x^0 , x_{i^0}) \leq \delta_{i^0}$$

so that $x_{ik} \in B[x_{i^0} , \delta_{i^0}]$. We have then

$$0 = \langle j_{x_{i^0}} , g(x^0) - x^0\rangle = \sum_{k=1}^{s} p_{ik}(x^0) \langle j_{x_{i^0}} , y_{ik} - x^0\rangle = \sum_{k=1}^{s} p_{ik}(x^0) \langle j_{x_{i^0}} , f(x_{ik}) - x^0\rangle =$$

$$= \sum_{k=1}^{s} p_{ik}(x^0) \langle j_{x_{i^0}} , f(x_{ik}) - x_{i^0}\rangle + \langle j_{x_{i^0}} , x_{i^0} - x^0\rangle \geq$$

$$\geq 2/3 \, |f(x_{i^0}) - x_{i^0}|^2 - 1/3 \, |f(x_{i^0}) - x_{i^0}|^2 = 1/3 \, |f(x_{i^0}) - x_{i^0}|^2 > 0 .$$

Therefore we have reached a contradiction and the Theorem is proved.

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