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Density and continuous functions. Nota II

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Density and continuous functions.* Nota II di OFELIA TERESA ALAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Relazione tra un tipo particolare di prodotto di spazi topologici e le funzioni θ–continue. Dimostrazione di un risultato generalizzante un teorema di A.M. Gleason.

In [1] we have studied the relations between the density and the pseudocaliber of a particular type of product of topological spaces and the continuous functions defined in these spaces. (To be more specific, we have dealt there with maximal sets of elementary disjoint open sets; for instance, in the lemma and theorems 4, 5 and 6). In the present paper we continue this study, considering θ–continuous functions and the pseudoweight at each point of a topological space. Our main theorem is a generalization of a well-known theorem of A. M. Gleason ([3] or [7], p. 38).

1. PRELIMINARIES AND NOTATIONS

For any set Z, $|Z|$ denotes the cardinal number of Z.

For any topological space E and any subset A of E, $\bar{A}$ denotes the closure of A in E.

DEFINITION ([4]). Let E and F be two topological spaces and $f: E \to F$ be a function. The function f is θ–continuous if for any $x \in E$ and any open neighborhood V of $f(x)$, the inverse set $f^{-1}(\bar{V})$ is a neighborhood of x.

DEFINITION ([2]). Let E be a topological space and $x \in E$, the pseudo weight of E at the point x is the least power of a family of open sets whose intersection is equal to $\{x\}$.

(*) Nella seduta del 13 giugno 1970.

Let q and m be two infinite cardinal numbers with $m^q = m$. Let T be a nonempty set and, for each $t \in T$, (X_t, σ_t) be a nonempty Hausdorff space with density less than or equal to m. In $X = \prod_{t \in T} X_t$ we consider the topology σ generated by the set of the subsets V of X, $V = \prod_{t \in T} V_t$, such that $V_t \in \sigma_t$, $\forall t \in T$ and $|\{t \in T \mid V_t \neq X_t\}| \leq q$. (A set like this one will be called an elementary open set). We fix a point $(e_t)_{t \in T} \in X$.

For any subset P of T, $P \neq \varnothing$, we denote by P^* the projection of X onto $\prod_{t \in P} X_t$.

For any $d \in X$, $d = (d_t)_{t \in T}$, and for any T_1, subset of T, we shall denote by $x(d; T_1)$ the point $(x_t)_{t \in T}$ of X, where $x_t = d_t$ if $t \in T_1$ and $x_t = e_t$ if $t \in T - T_1$.

For any infinite cardinal number r we put

$$F(r) = \{x \in X \mid |\{t \in T \mid x_t \neq e_t\}| < r\} \quad , \quad \text{where } x = (x_t)_{t \in T}.$$

In $F(r)$ we shall consider the topology induced by σ. (Notice that $F(r) = X$ if $r > |T|$).

THEOREM 1 ([1]). *If* $|T| \leq 2^m$ *the density of* X *does not exceed* m.

2. MAIN RESULTS

First we shall prove a theorem which is an immediate consequence of theorem 1.

THEOREM 2. *For any infinite cardinal* r, *if* $|T| = m$, *then* $F(r)$ *has density less than or equal to* m.

Proof. Suppose $|T| = m$. Let D be a dense subset of X with $|D| \leq m$ (it exists by theorem 1).

First let us suppose that $r \leq q$. The set M of all points $x(d; T_1)$ where d changes in D and T_1 changes in the set of all subsets of T whose cardinality is less than r, has cardinality $|M|$ less than or equal to m (because $m^q = m$). Furthermore, M is dense in $F(r)$. Indeed, let $V = \prod_{t \in T} V_t$ be an elementary open set in X such that $V \cap F(r) \neq \varnothing$. It follows that the set $T_1 = \{t \in T \mid e_t \notin V_t, V_t \neq X_t\}$ has cardinality less than r. If $d \in V \cap D$, then $x(d; T_1) \in V \cap F(r)$.

Now, let us suppose that $q < r$. The set M of all points $x(d; T_1)$ where d changes in D and T_1 changes in the set of all subsets of T whose cardinality does not exceed q, has cardinality $|M|$ less than or equal to m and it is dense in $F(r)$.

The proof of the next theorem follows that of the theorem of A. M. Gleason ([7], p. 38). Let r be an infinite cardinal number.

THEOREM 3. *Let* Y *be a Hausdorff space such that the pseudo-weight of* Y *at any point does not exceed* m. *If* $f: F(r) \to Y$ *is continuous, there is a subset* P *of* T, *with* $|P| \leq m$, *such that if* $x, y \in F(r)$ *and* $P^*(x) = P^*(y)$, *then* $f(x) = f(y)$.

Proof. Since for $|T| \leq m$ the result is trivial, let us suppose that $|T| > m$. Let q' be the cardinal number successor of q (thus $q' \leq m$), μ be the first ordinal of cardinality q' and H be the set of all ordinals smaller (in the usual order over the ordinals) than μ. Thus $|H| = q'$.

To simplify the notation, instead of $F(r)$ we use F. For any $x \in F$ we fix a subset $T(x)$ of T, with $|T(x)| \leq m$, such that if $y = (y_t)_{t \in T}$ belongs to F and $x_t = y_t$, $\forall t \in T(x)$, then $f(y) = f(x)$. It exists because the inverse set $f^{-1}(\{f(x)\})$ is the intersection of at most m open sets in X, hence it contains the intersection of m elementary open sets in X, neighborhoods of x.

We shall construct a family $(Z_\alpha)_{\alpha \in H}$ of subsets of F such that: 1) $|Z_\alpha| \leq m$, $\forall \alpha \in H$; 2) for any $\alpha \in H$, $\overline{T^*_\alpha(Z_\beta)} = T^*_\alpha(F)$, where β is the ordinal successor of α and $T_\alpha = \cup \{T(x) \mid x \in Z_\gamma, \gamma \leq \alpha\}$. (Here, $\leq$ is the order over the ordinals).

For, if there is such a family, the set $P = \bigcup_{\alpha \in H} T_\alpha$ verifies the thesis. On the contrary, let us suppose that there are $x = (x_t)_{t \in T}$, $y = (y_t)_{t \in T}$ belonging to F, with $P^*(x) = P^*(y)$ and $f(x) \neq f(y)$. Then there are two disjoint open sets in Y, U and U', and two elementary open sets $V = \prod_{t \in T} V_t$ and $W = \prod_{t \in T} W_t$ such that $V_t = W_t$, $\forall t \in P$, $x \in V \cap F \subset f^{-1}(U)$ and $y \in W \cap \cap F \subset f^{-1}(U')$. There is $\alpha \in H$ such that the set $P_0 = \{t \in P \mid V_t \neq X_t\} \subset T_\alpha$ (because $|P_0| \leq q$). Let β be the ordinal successor of α and $z = (z_t)_{t \in T}$ an element of Z_β, with $T^*_\alpha(z) \in \prod_{t \in T_\alpha} V_t$. Putting $a = (a_t)_{t \in T}$, $b = (b_t)_{t \in T}$, where $a_t = b_t = z_t$ for any $t \in P$, $a_t = x_t$ and $y_t = b_t$ if $t \in T - P$, we have $f(a) = f(z) = f(b)$, because $T(z) \subset P$. But $f(a) \in U$ and $f(b) \in U'$, which is impossible. The proof is completed, since we may construct that family as follows: let $x_0 \in F$ and put $Z_0 = \{x_0\}$; let δ be an element of H, $\delta \neq 0$, and suppose that we have the sets Z_α for any ordinal α preceding δ, verifying $|Z_\alpha| \leq m$ and if β is the ordinal successor of α and β precedes δ, then $\overline{T^*_\alpha(Z_\beta)} = T^*_\alpha(F)$, as in condition 2). We shall construct Z_δ. The cardinality of the set $K = \cup \{T(x) \mid x \in Z_\alpha, \alpha < \delta\}$ does not exceed m; so let Z be a set of cardinality not greater than m, dense in $K^*(F)$. Hence we put $Z_\delta = \{x = (x_t)_{t \in T} \mid K^*(x) \in Z$ and $x_t = e_t$, $\forall t \notin K\}$.

COROLLARY 1. *Let* Y *be a Hausdorff space such that the weight of* Y *at any point does not exceed m. If* $f : F(r) \to Y$ *is* θ*-continuous, there is a subset* P *of* T, *with* $|P| \leq m$, *such that if* x, y *belong to* $F(r)$ *and* $P^*(x) = P^*(y)$, *then* $f(x) = f(y)$.

Proof. Let τ be the topology in Y and let us denote by τ_0 the topology in Y generated by the set $\{\overset{\circ}{\overline{U}} \mid U \in \tau\}$, where $\overset{\circ}{\overline{U}}$ is the interior of the closure of U (in the topology τ). Considering the topology τ_0 in Y, the function f is continuous and we apply theorem 3.

THEOREM 4. *Let* Y *be a Hausdorff space such that the pseudo-weight of* Y *at any point does not exceed* 2^m. *If* $f : X \to Y$ *is continuous, there is a subset* Z *of* X, *with* $|Z| \leq m$, *such that* $f(Z)$ *is dense in* $f(X)$.

Proof. Since $(2^m)^q = 2^m$, it follows, from theorem 3, that there is a subset P of T, with $|P| \leq 2^m$, such that if x, y belong to X and $P^*(x) = P^*(y)$, then $f(x) = f(y)$. So, it is sufficient to prove the theorem for $|T| \leq 2^m$.

Suppose $|T| \leq 2^m$. Let Z be a dense subset of X, with $|Z| \leq m$ (it exists by virtue of theorem 1). The image set $f(Z)$ is dense in $f(X)$ because f is continuous.

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