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A generalization of the l_1-algebra of a commutative semi-group

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Algebra. — *A generalization of the l_1–algebra of a commutative semi-group.* Nota di OLUSOLA AKINYẸLE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Dato un semigruppo commutativo discreto S, e un'algebra di Banach commutativa A con o senza identità, consideriamo il convoluto in algebra di Banach $l_1(S, A)$, consistente di tutte le funzioni f definite in S con valori di A tali che $\sum_{s \in S} \|f(s)\|_A$ sia finito. Nel lavoro $\hat{S}$ indica l'insieme di tutti gli ideali massimi regolari dell'algebra A. Nella 3 parte abbiamo dimostrato il seguente teorema: per ogni f e $l_1(S, A)$ si definisca la « trasformata » di f rispetto a un punto fisso (M, χ) di $\mathfrak{M}(A) \times \hat{S}$ come

$$J_{(M,\chi)}(f) = \sum_{s \in S} \varphi_m(f(s)) \chi(s)$$

dove φ_m è un omomorfismo continuo di A sui numeri complessi. Allora $J_{(M,\chi)}$ è un omomorfismo continuo non nullo di $l_1(S, A)$. Reciprocamente, dato un omomorfismo continuo non nullo h su $l_1(S, A)$, $\exists (M, \chi) \in \mathfrak{M}(A) \times \hat{S}$, tale che per $f \in l_1(S, A)$, $h(f) = J_{(M,\chi)}(f)$. Nella quarta parte, sotto le condizioni che S ha la proprietà che $xy = x^2 = y^2$ comporta che $x = y$ per $x, y \in S$, otteniamo una condizione necessaria e sufficiente perché $l_1(S, A)$ sia semisemplice.

§ 1. INTRODUCTION

Let G be a locally compact Abelian group and let $L'(G)$ denote the algebra, under convolution, of all absolutely integrable complex-valued functions on G. It is known [4, Theorem 34 B] that there exists a one-to-one correspondence between the regular maximal ideals of $L'(G)$ and the set of all characters of the group G.

Suppose S is a discrete commutative semigroup. Hewitt and Zuckerman in [3] have considered the convolution Banach algebra $l_1(S)$ defined as the set of all complex-valued functions f on S that vanish except on a countable subset of S and for which $\|f\| = \sum_{x \in S} |f(x)|$ is finite. They show that there exists a one-to-one correspondence between the space of regular maximal ideals of $l_1(S)$ and the set of all semicharacters of the semigroup S.

Hausner in [2] has extended the result for $L'(G)$ to the algebra $B'(G, A)$ consisting of all of Bochner integrable functions defined on a locally compact Abelian group with values in an arbitrary Banach algebra A. The work in this paper was motivated by the results of Hausner and that of Hewitt and Zuckerman. Throughout this paper it will be assumed that S is a discrete commutative semigroup and that A is a commutative Banach algebra with or without an identity. We consider the convolution algebra $l_1(S, A)$ which consists of all functions f defined on S with values in A such that $\sum_{s \in S} \|f(s)\|_A$ is finite. In Section 3, we discuss the complex homomorphisms on $l_1(S, A)$ and obtain a result analogous to that obtained by Hausner for $B'(G, A)$.

(*) Nella seduta del 13 giugno 1970.

Furthermore in Section 4, we obtain a necessary and sufficient condition for $l_1(S, A)$ to be semi-simple.

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§ 2. PRELIMINARIES

Let S be a semigroup, and A a Banach algebra. Denote by $l_1(S, A)$ the set of all functions on S with values in A, that vanish except on a countable subset of S and for which

$$\|f\| = \sum_{s \in S} \|f(s)\|_A < \infty. \tag{1}$$

The sum here is taken to mean that f vanishes outside a countable set $\{s_1, s_2, \cdots, s_n, \cdots\}$ and $\sum_{n=1}^{\infty} \|f(s_n)\|_A$ is finite. The sum is independent of the arrangement of $\{s_1, s_2, \cdots s_n, \cdots\}$ since the sum is absolutely convergent.

If we define addition and scalar multiplication as follows:

$$(f_1 + f_2)(s) = f_1(s) + f_2(s),$$

$$(\alpha f_1)(s) = \alpha \cdot f_1(s) \qquad \text{for all} \quad s \in S \text{ and } \alpha$$

a complex number, then $l_1(S, A)$ becomes a complex linear space.

With the norm in $l_1(S, A)$ defined by (1), $l_1(S, A)$ becomes a complete normed linear space.

For $f, g \in l_1(S, A)$ let product be defined by convolution as follows:

$$f * g(s) = \sum_{u,v,uv=s \in S} f(u) g(v).$$

LEMMA 2.1. *For* $f, g \in l_1(S, A)$, $f * g \in l_1(S, A)$ *and* $\|f * g\| \leq \|f\| \|g\|$. *The proof is straightforward.*

It is easy to verify that with product defined as above, $l_1(S, A)$ is an algebra. Furthermore if S and A are both commutative, then $l_1(S, A)$ is commutative and $l_1(S, A)$ becomes a complex commutative Banach algebra, which specializes to $l_1(S)$ when A is taken to be the complex numbers.

DEFINITION 2.2. A function f on S to A is simple if it is a constant on each of a finite number of subsets E_j of S and equal to zero on $S \sim \bigcup_{j=1}^{n} E_j$. A function f is a countably-valued function if it assumes at most a countable set of values in A, assuming each value different from zero on a subset E_j of S.

LEMMA 2.3. *The simple functions are dense in* $l_1(S, A)$.

Proof. Given $f \in l_1(S, A)$, f is a countably-valued function.

Suppose f takes non-zero values $\{a_1, a_2, \cdots, a_k, \cdots\}$ on $\{E_1, E_2, E_3, \cdots \cdots E_k \cdots\}$ where

$$E_k = \{s \in S : f(s) = a_k\}.$$

Define a function g_n on S by setting

$g_n = \sum_{k=1}^{n} a_k \xi_{E_k}$ where ξ_{E_k} is the characteristic function of the subset E_k of S. For each finite n, g_n is a simple function in $l_1(S, A)$. We will now show that $g_n \to f$ as $n \to \infty$. Let $s \in S$, then $f(s)$ is either 0 or s belongs to one of the sets $\{E_1, E_2, \cdots, E_k, \cdots\}$ where $f(s) \neq 0$. Suppose $f(s) = 0$, then $\|f(s) - g_n(s)\| = 0$. Suppose $f(s) \neq 0$, then $s \in E_k$ for some integer k. Set $N_k = k$ then

$$\|f(s) - g_n(s)\| = 0 \qquad \text{provided} \quad n \geq N_k.$$

Thus

$$(2.3.1) \qquad \lim_{n\to\infty} \|f(s) - g_n(s)\| = 0 \qquad \text{for each} \quad s \in S.$$

Now $\sum_{s \in S} \|f(s) - g_n(s)\| < \infty$ for each fixed n, since $f, g_n \in l_1(S, A)$. It follows that the series $\sum_{k=1}^{\infty} \|f(s_k) - g_n(s_k)\|$ of non-negative real numbers is convergent for each fixed n.

But $\|f(s) - g_n(s)\| \leq \|f(s)\|$ for each $s \in S$ and for all n. This means that $\sum_{k=1}^{\infty} \|f(s_k) - g_n(s_k)\|$ is convergent for all n and hence the sequence of partial sums $J_{(m,n)} = \sum_{k=1}^{m} \|f(s_k) - g_n(s_k)\|$ for finite m and n, converges as $(m, n) \to \infty$. By a double limit theorem [1; p 371, Theorem 12–39] we have

$$\lim_{(m,n)\to\infty} J_{(m,n)} = \lim_{n\to\infty} \left(\lim_{m\to\infty} J_{(m,n)} \right).$$

A similar result holds if we interchange the roles of m and n; i.e.

$$\lim_{(m,n)\to\infty} J_{(m,n)} = \lim_{m\to\infty} \left(\lim_{n\to\infty} J_{(m,n)} \right).$$

Hence we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \|f(s_k) - g_n(s_k)\| = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=1}^{m} \|f(s_k) - g_n(s_k)\|$$

$$= \lim_{m\to\infty} \sum_{k=1}^{m} \lim_{n\to\infty} \|f(s_k) - g_n(s_k)\|$$

$$= \sum_{k=1}^{\infty} \lim_{n\to\infty} \|f(s_k) - g_n(s_k)\|.$$

This $\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} \|f(s_k) - g_n(s_k)\| = \sum_{k=1}^{\infty} \lim_{n\to\infty} \|f(s_k) - g_n(s_k)\|$

and $\lim\limits_{n\to\infty} \sum\limits_{s\in S} \|f(s) - g_n(s)\| = \sum\limits_{s\in S} \lim\limits_{n\to\infty} \|f(s) - g_n(s)\|$

$= 0$ by (2.3.1).

Hence $\lim\limits_{n\to\infty} \|f - g_n\| = 0$, and the proof is complete.

REMARK 1. Define a function af on S by setting $af(s) = a\cdot f(s)$ for $a \in A$ and $f \in l_1(S)$. The simple functions are of the form $g_n = \sum\limits_{k=1}^{n} a_k \xi_{E_k}$ where ξ_{E_k} is the characteristic function of the subset E_k of S and $a_k \in A$. Denote by P the set of all finite linear combinations of af, $a \in A$, $f \in l_1(S)$. It is easy to see that the set P contains the simple functions, and hence dense in $l_1(S, A)$, since the simple functions are dense in $l_1(S, A)$.

§ 3. COMPLEX HOMOMORPHISMS OF $l_1(S, A)$

DEFINITION 3.1. A multiplicative function on a semigroup S is any complex function τ on S satisfying the functional equation $\tau(xy) = \tau(x)\tau(y)$ for all $x, y \in S$. A multiplicative function is called a semicharacter if it is different from 0 at some point and is bounded. If χ is a semicharacter on S, then it can be easily shown that $|\chi(x)| \leq |\tau(x)|$ for all $x \in S$. Furthermore a semicharacter of a group is a character.

DEFINITION 3.2. Let $\hat{S}$ be the set of all semicharacters of S and $\mathfrak{M}(A)$ the set of all maximal regular ideals of the Banach algebra A with or without a unit element. Denote by $\mathfrak{M}(A) \times \hat{S}$ the Cartesian product of $\mathfrak{M}(A)$ and $\hat{S}$, i.e. the set of pairs (M, χ) with $M \in \mathfrak{M}(A)$ and $\chi \in \hat{S}$. The following is our main theorem.

THEOREM 3.3. *Let* $\hat{S}$ *and* $\mathfrak{M}(A)$ *be as defined above. For any* $f \in l_1(S, A)$ *define the 'transform' of* f *with respect to a fixed point* (M, χ) *of* $\mathfrak{M}(A) \times \hat{S}$ *by*

$$J_{(M,\chi)}(f) = \sum_{s\in S} \varphi_M(f(s))\,\chi(s)$$

where φ_M *is a continuous homomorphism of* A *onto the complex numbers.* Then $J_{(M,\chi)}$ *is a non-zero continuous homomorphism on* $l_1(S, A)$. *Conversely given a non-zero continuous homomorphism* h *on* $l_1(S, A)$, $\exists\,(M, \chi) \in \mathfrak{M}(A) \times \hat{S}$ *such that for* $f \in l_1(S, A)$, $h(f) = J_{(M,\chi)}(f)$.

[Note: The series $\sum\limits_{s\in S} \varphi_M(f(s))\,\chi(s)$ is absolutely convergent].

Proof. That $J_{(M,\chi)}$ is a non-zero continuous homomorphism on $l_1(S, A)$ can be shown easily and we omit the proof. Now let h be a non-zero continuous homomorphism on $l_1(S, A)$. Let $f \in l_1(S, A) \ni h(f) \neq 0$. For any $g \in l_1(S, A)$ and $a \in A$, the function ag defined on S by setting $ag(s) = a\cdot g(s)$ is an element of $l_1(S, A)$. Let $f \in l_1(S, A) \ni h(f) \neq 0$ and consider $\frac{h(af)}{h(f)}$. It can be easily shown that $\frac{h(af)}{h(f)}$ is independent of the choice of $f \in l_1(S, A)$. In view of this we know that $\frac{h(af)}{h(f)}$ depends only on a. We now define a function φ_h on A onto the complex numbers by setting $\varphi_h(a) = \frac{h(af)}{h(f)}$, for $a \in A$. Assume temporarily that A has a unit element e. Then $\varphi_h(e) = 1$ and $\varphi_h \neq 0$. Further φ_h is easily seen to be well-defined, continuous and linear. Now for $a_1, a_2 \in A$.

$$\varphi_h(a_1 a_2) = \frac{h(a_1 a_2 f)}{h(f)} = \frac{h(a_1 a_2 f)}{h(f)} \cdot \frac{h(f)}{h(f)} = \frac{h(a_1 f)}{h(f)} \cdot \frac{h(a_2 f)}{h(f)} = \varphi_h(a_1)\,\varphi_h(a_2),$$

so that φ_h is multiplicative.

Thus φ_h is a non-zero continuous homorphism of A onto the complex numbers, and as is well known there exists $M \in \mathfrak{M}(A)$ (depending only on h) $\ni$

$$\varphi_h(a) = \varphi_M(a).$$

Finally let $a \in A$ and $g \in l_1(S)$, then

$$h(ag * f) = h(aeg * f) = h(eg * af) = h(eg)\,h(af).$$

Hence $h(ag)\,h(f) = h(eg)\,h(af)$

$$\text{(3.3.1)} \qquad \text{and} \quad h(ag) = h(eg)\,\varphi_M(a) \quad \text{for} \quad g \in l_1(S).$$

But the set $el_1(S) = \{eg \in l_1(S, A) : g \in l_1(S)\}$ is isomorphic and isometric with $l_1(S)$. Since h is a non-zero continuous homomorphism on $l_1(S, A)$, it is not zero identically on $el_1(S)$ in view of (3.3.1) and because linear combinations of functions ag with $g \in l_1(S)$ $a \in A$ are dense in $l_1(S, A)$ (see Remark 1). It follows that h is a non-zero continuous homomorphism on $el_1(S)$. By the isometric isomorphism of $el_1(S)$ with $l_1(S)$ it follows that $\exists \chi \in \hat{S} \ni$

$$h(eg) = \sum_{s \in S} g(s)\,\chi(s) \quad \text{for all} \quad g \in l_1(S) \quad \text{and then}$$

$$\text{(3.3.2)} \qquad h(ag) = \varphi_M(a)\,h(eg) = \sum_{s \in S} \varphi_M(ag(s))\,\chi(s) = J_{(M,\chi)}(ag)$$

Suppose $f \in l_1(S, A)$, then $\exists$ a sequence of simple functions

$$\{g_n\} \subset l_1(S, A) \ni g_n \to f \quad \text{as} \quad n \to \infty. \quad \text{By (3.3.2)} \quad h(g_n) = J_{(M,\chi)}(g_n).$$

Hence $$h(f) = J_{(M,\chi)}(f).$$

Suppose now that A has no unit element; then by Theorem 20 c (page 59) of [4] we can embed A isometrically and isomorphically in a Banach algebra A^* with unit e in such a way that the maximal ideals in A^* are the maximal regular ideals of A and A itself. The homomorphisms of A^* onto the complex numbers are the φ_M ($M \in \mathfrak{M}(A)$) and φ_A where $\varphi_A(a + \lambda e) = \lambda$, for $a \in A$ and λ a complex number. By what has been proved, the non-zero continuous homomorphism on $l_1(S, A^*)$ are the $J_{(M,\chi)}$ and additional functionals

$J_{(A,\chi)}$. But for $f \in l_1(S, A)$, $J_{(A,\chi)}(f) = \sum_{s \in S} \varphi_A(f(s))\,\chi(s) = 0$. However, the functionals $J_{(A,\chi)}$ are identically zero on $l_1(S, A)$ and thus the non-zero continuous homomorphisms are the $J_{(M,\chi)}$, which completes the proof.

COROLLARY 3.4. *There is a one-to-one correspondence between the points of* $\mathfrak{M}(l_1(S, A))$, *that is the space of maximal regular ideals of* $l_1(S, A)$ *and* $\mathfrak{M}(A) \times \hat{S}$.

§ 4. THE RADICAL AND SEMISIMPLICITY OF $l_1(S, A)$

THEOREM 4.1. *Let* S *be a discrete commutative semigroup with the property that* $xy = x^2 = y^2$ *implies that* $x = y$ *for* $x, y \in S$. *Then a necessary and sufficient condition that a function* f *be in the radical of* $l_1(S, A)$ *is that the range of* f *be in the radical of* A.

Proof. Let R be the radical of A, then $R = \bigcap_{M \in \mathfrak{M}(A)} M$. Suppose $f \in l_1(S, A)$ has its range in R; then $J_{(M,\chi)}(f) = 0$, for all $(M, \chi) \in \mathfrak{M}(A) \times \hat{S}$. It follows then that f belongs to the radical of $l_1(S, A)$.

Suppose f belongs to the radical of $l_1(S, A)$, then $J_{(M,\chi)}(f) = 0$ for every $(M, \chi) \in \mathfrak{M}(A) \times \hat{S}$. This implies $\sum_{s \in S} \varphi_M(f(s))\,\chi(s) = 0$ for arbitrary $(M, \chi) \in \mathfrak{M}(A) \times \hat{S}$. If we define a function $\varphi_M f$ on S by setting $\varphi_M f(s) = \varphi_M(f(s))$, then $\varphi_M f \in l_1(s)$. Also $\varphi_M f$ belongs to the radical of $l_1(s)$ [3, Theorem 2.8]. By the assumption on S, $l_1(S)$ is semisimple [3, Theorem 5.8]. Hence $\varphi_M f(s) = 0$ for an arbitrary $M \in \mathfrak{M}(A)$. It follows that f has its range in R.

The following corollary follows readily from the last theorem.

COROLLARY 4.2. *If* S *is as in Theorem* 4.1, *then* $l_1(S, A)$ *is semisimple if and only if* A *is semisimple.*

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