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On Almansi-Michell's problem for orthotropic beams

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Meccanica. — *On Almansi–Michell's problem for orthotropic beams.* Nota di CONSTANTIN I. BORȘ, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — Si da una nuova soluzione per il problema di Almansi–Michell definita dalle (4) e (5) per un cilindro anisotropo. La soluzione indicata in questa Nota è più semplice ed ha alcuni vantaggi rispetto a quelle fin'ora conosciute.

We will consider an orthotropic beam limited by two planes $x_3 = 0$, $x_3 = h$ and by a cylindrical surface $\mathcal{F}$.

The domain occupied by the beam will be denoted by $\mathcal{D}$, the domain of a cross-section of the beam and its area will be denoted by S and the boundary of S by Γ.

We will take the axes of $x_1\,x_2\,x_3$ in such a way that

$$\iint_S x_1 \, d\sigma = 0 , \qquad \iint_S x_2 \, d\sigma = 0 , \qquad \iint_S x_1\, x_2 \, d\sigma = 0 . \tag{1}$$

These mean that the axis of x_3 is the central-line of the beam and the axes of x_1 and x_2 are the principal axes of inertia of the end at $x_3 = 0$.

The beam is supposed to be orthotropic so that the relations between the components of strain γ_{ij} and the components of stress σ_{ij} may be written in the form [2]:

$$\left\{ \begin{aligned} \gamma_{11} &= \frac{1}{E} (\nu_{11}\,\sigma_{11} + \nu_{12}\,\sigma_{22} - \nu_1\,\sigma_{33}) , \\ \gamma_{22} &= \frac{1}{E} (\nu_{12}\,\sigma_{11} + \nu_{22}\,\sigma_{22} - \nu_2\,\sigma_{33}) , \\ \gamma_{33} &= \frac{1}{E} (-\nu_1\,\sigma_{11} - \nu_2\,\sigma_{22} + \sigma_{33}) , \\ \gamma_{12} &= \frac{1}{D}\,\sigma_{12} \quad , \quad \gamma_{23} = \frac{1}{L}\,\sigma_{23} \quad , \quad \gamma_{31} = \frac{1}{M}\,\sigma_{31} , \end{aligned} \right. \tag{2}$$

where ν_{ij} , ν_i , E , D , L , M are constants which characterize the elastic qualities of the material of the beam.

The strain components γ_{ij} are connected with the components of displacement u_i by

$$\left\{ \begin{aligned} \gamma_{ii} &= u_{i,i} && \text{(not summed)}, \\ \gamma_{ij} &= u_{i,j} + u_{j,i} \, . && (i \neq j) \end{aligned} \right. \tag{3}$$

and they must satisfy the compatibility conditions of Saint-Venant.

(*) Nella seduta del 13 giugno 1970.

If we suppose that there are no body forces the components of stress satisfy the equilibrium equations given by

$$\sigma_{ij,j} = 0 \quad \text{in} \quad \mathcal{D}\ ^{(1)}. \tag{4}$$

We will suppose that the tractions applied on the lateral surface are such that

$$\sigma_{ij}\, n_j = \tau_i (x_1 , x_2) \qquad \text{on} \quad \Gamma, \tag{5}$$

and at the ends are applied tractions in order to equilibrate the loads (5). In the formulae (5) n_i are the direction cosines of the exterior normal to the surface $\mathcal{F}$ and $\tau_i (x_1 , x_2)$ are given functions of x_1 and x_2.

In order to solve this problem, first we will find a solution which satisfies the equations (4), (5), and after that it remains to satisfy the end conditions: this is another problem and we know how to solve it.

Two ways of finding stresses which satisfy (4) and (5) were given, for the isotropic case, 70 years ago, by Almansi [1] and Michell [6].

A few years ago, Khatiashvili has generalized the results of Almansi to the orthotropic case and to the anisotropic case with one plane of elastic symmetry and he called the problem definite by (4) and (5)–Almansi-Michell's problem–[3], [4], [5].

With a view to pointing out the importance of this problem we should like to remember only that many problems about non-cylindrical beams may be reduced to it (for example the problem of torsion and the problem of bending by a transverse load concerning naturally slightly curved beams and naturally twisted beams, etc. [2]).

There are many ways to solve this problem. Almansi himself has pointed out the fact that there are infinite means to satisfy the equations (4) and (5).

Now, in this Note we try to give a more simple solution for this problem, in the orthotropic case, which will take out some advantages by comparison with others.

For this purpose let us suppose that the components of stress are given by the following formulae:

$$\tag{6} \begin{cases} \sigma_{11} = \dfrac{\partial^2 \Phi}{\partial x_2^2} - \mathrm{M}(\omega_1 + \theta_1) \ , \quad \sigma_{22} = \dfrac{\partial^2 \Phi}{\partial x_1^2} - \mathrm{L}(\omega_1 + \theta_2) \ , \quad \sigma_{12} = - \dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} , \\ \sigma_{33} = \Omega + \mathrm{E A} x_3 + \dfrac{1}{2} \mathrm{E} \left(\dfrac{b_1}{\nu_2} x_1 + \dfrac{a_1}{\nu_1} x_2 \right) x_3^2 , \\ \sigma_{23} = \mathrm{L} \left(\dfrac{\partial \omega_1}{\partial x_2} + \dfrac{\partial \theta_2}{\partial x_2} \right) x_3 + \mathrm{L} \dfrac{\partial \omega_0}{\partial x_2} , \\ \sigma_{31} = \mathrm{M} \left(\dfrac{\partial \omega_1}{\partial x_1} + \dfrac{\partial \theta_1}{\partial x_1} \right) x_3 + \mathrm{M} \dfrac{\partial \omega_0}{\partial x_1} , \end{cases}$$

(1) The index j after comma indicates partial differentiation with respect to x_j. We use also the summation convention over the repeated indices.

where ω_0, ω_1, Φ are unknown functions of x_1 and x_2,

$$(7)\quad \begin{cases} \Omega = \mathrm{E}\left[\omega_1 + \frac{1}{2}\,\theta + \frac{1}{6}\left(\frac{\nu_1}{\nu_2}\,b_1\,x_1^3 + \frac{\nu_2}{\nu_1}\,a_1\,x_2^3\right)\right] + \\ \qquad + \nu_1\left[\frac{\partial^2\Phi}{\partial x_2^2} - \mathrm{M}\,(\omega_1+\theta_1)\right] + \nu_2\left[\frac{\partial^2\Phi}{\partial x_1^2} - \mathrm{L}\,(\omega_1+\theta_2)\right], \\ \theta_1 = b_1\,x_1\,x_2^2 - c_1\,x_1\,x_2 \quad , \quad \theta_2 = a_1\,x_1^2\,x_2 + c_1\,x_1\,x_2 \quad , \quad \theta = \theta_1 + \theta_2, \end{cases}$$

A, a_1, b_1, c_1 being constants which must be determined in such a way as to ensure the existence of the functions ω_0, ω_1 and Φ.

From the equations of equilibrium we find that the functions ω_0 and ω_1 must satisfy the equations

$$(8)\qquad \mathrm{M}\,\frac{\partial^2\omega_0}{\partial x_1^2} + \mathrm{L}\,\frac{\partial^2\omega_0}{\partial x_2^2} + \mathrm{EA} = 0 \qquad \text{in S},$$

and

$$(9)\qquad \mathrm{M}\,\frac{\partial^2\omega_1}{\partial x_1^2} + \mathrm{L}\,\frac{\partial^2\omega_1}{\partial x_2^2} + \mathrm{E}\left(\frac{b_1}{\nu_2}\,x_1 + \frac{a_1}{\nu_1}\,x_2\right) = 0 \qquad \text{in S}.$$

The deformations corresponding to the stresses (6) are given by

$$\gamma_{11} = \frac{1}{\mathrm{E}}\left\{\nu_{11}\left[\frac{\partial^2\Phi}{\partial x_2^2} - \mathrm{M}(\omega_1+\theta_1)\right] + \nu_{12}\left[\frac{\partial^2\Phi}{\partial x_1^2} - \mathrm{L}\,(\omega_1+\theta_2)\right] - \nu_1\,\Omega\right\} -$$

$$- \mathrm{A}\nu_1\,x_3 - \frac{1}{2}\left(\frac{\nu_1}{\nu_2}\,b_1\,x_1 + a_1\,x_2\right)x_3^2,$$

$$\gamma_{22} = \frac{1}{\mathrm{E}}\left\{\nu_{12}\left[\frac{\partial^2\Phi}{\partial x_2^2} - \mathrm{M}\,(\omega_1+\theta_1)\right] + \nu_{22}\left[\frac{\partial^2\Phi}{\partial x_1^2} - \mathrm{L}\,(\omega_1+\theta_2)\right] - \nu_2\,\Omega\right\} -$$

$$- \mathrm{A}\nu_2\,x_3 - \frac{1}{2}\left(b_1\,x_1 + \frac{\nu_2}{\nu_1}\,a_1\,x_2\right)x_3^2,$$

$$\gamma_{33} = \omega_1 + \frac{1}{2}\,\theta + \frac{1}{6}\left(\frac{\nu_1}{\nu_2}\,b_1\,x_1^3 + \frac{\nu_2}{\nu_1}\,a_1\,x_2^3\right) + \mathrm{A}x_3 + \frac{1}{2}\left(\frac{b_1}{\nu_2}\,x_1 + \frac{a_1}{\nu_1}\,x_2\right)x_3^2,$$

$$\gamma_{12} = -\frac{1}{\mathrm{D}}\,\frac{\partial^2\Phi}{\partial x_1\,\partial x_2},$$

$$\gamma_{23} = \left(\frac{\partial\omega_1}{\partial x_2} + \frac{\partial\theta_2}{\partial x_2}\right)x_3 + \frac{\partial\omega_0}{\partial x_2},$$

$$\gamma_{31} = \left(\frac{\partial\omega_1}{\partial x_1} + \frac{\partial\theta_1}{\partial x_1}\right)x_3 + \frac{\partial\omega_0}{\partial x_1}$$

and they will satisfy the conditions of compatibility if the function Φ satisfies the equation

$$(10)\qquad \beta_{22}\,\frac{\partial^4\Phi}{\partial x_1^4} + (2\,\beta_{12} + \beta_{33})\,\frac{\partial^4\Phi}{\partial x_1^2\,\partial x_2^2} + \beta_{11}\,\frac{\partial^4\Phi}{\partial x_2^4} =$$

$$= (\mathrm{L}\beta_{22} + \mathrm{M}\beta_{12} + \nu_2)\,\frac{\partial^2\omega_1}{\partial x_1^2} + (\mathrm{L}\beta_{12} + \mathrm{M}\beta_{11} + \nu_1)\,\frac{\partial^2\omega_1}{\partial x_2^2} +$$

$$+ 2\,[(\mathrm{M}\beta_{11} + \nu_1)\,b_1\,x_1 + (\mathrm{L}\beta_{22} + \nu_2)\,a_1\,x_2] \qquad \text{in S},$$

where

(11) $$\beta_{ij} = \frac{\nu_{ij} - \nu_i \nu_j}{E} \qquad (i\,,j = 1\,,2) \quad , \quad \beta_{33} = \frac{1}{D}\,.$$

The third boundary condition (5) will be satisfied if we take the functions ω_0 and ω_1 such that

(12) $$\mathfrak{D}\omega_0 = \tau_3 \qquad \text{on} \quad \Gamma,$$

and

(13) $$\mathfrak{D}\omega_1 = -\mathrm{M}\,(b_1 x_2^2 - c_1 x_2)\, n_1 - \mathrm{L}\,(a_1 x_1^2 + c_1 x_1)\, n_2 \qquad \text{on} \quad \Gamma,$$

where the operator $\mathfrak{D}$ in given by

(14) $$\mathfrak{D} = \mathrm{M} n_1 \frac{\partial}{\partial x_1} + \mathrm{L} n_2 \frac{\partial}{\partial x_2}\,.$$

In order to ensure the existence of the function ω_0 we must take

$$\mathrm{A} = \frac{1}{\mathrm{ES}} \int_\Gamma \tau_3\, ds\,.$$

It is easy to see that the function ω_1 exists for any constants $a_1\,,b_1\,,c_1$.

From the first two boundary conditions (5) we get

(15) $$\begin{cases} \dfrac{\partial^2 \Phi}{\partial x_2^2}\, n_1 - \dfrac{\partial^2 \Phi}{\partial x_1\, \partial x_2}\, n_2 = \mathrm{M}\,(\omega_1 + \theta_1)\, n_1 + \tau_1\,, \\[2ex] -\dfrac{\partial^2 \Phi}{\partial x_1\, \partial x_2}\, n_1 + \dfrac{\partial^2 \Phi}{\partial x_1^2}\, n_2 = \mathrm{L}\,(\omega_1 + \theta_2)\, n_2 + \tau_2 \qquad \text{on} \quad \Gamma. \end{cases}$$

From (15) is obvious that we may obtain the function Φ in a similar way to that corresponding to Airy's function for the plane problem of orthotropic bodies.

Making use of this, it follows that we can choose the constants $a_1\,,b_1\,,c_1$ in such a way as to ensure the existence of the function Φ.

We can not end before remarking that it is possible to find the constants $a_1\,,b_1\,,c_1$ without knowing the function ω_1.

In this respect we may take into account that

(16) $$\omega_1 = -\frac{1}{6}\,\mathrm{E}\left(\frac{b_1}{\mathrm{M}\nu_2}\, x_1^3 + \frac{a_1}{\mathrm{L}\nu_1}\, x_2^3\right) + \omega_1^*$$

is a solution of the equation (9) if ω_1^* verifies the equation

(17) $$\mathrm{M}\,\frac{\partial^2 \omega_1^*}{\partial x_1^2} + \mathrm{L}\,\frac{\partial^2 \omega_1^*}{\partial x_2^2} = 0 \qquad \text{in} \quad \mathrm{S}\,,$$

and the fact that

(18) $$\int_\Gamma (\omega \mathfrak{D}\varphi - \varphi \mathfrak{D}\omega)\, ds = 0$$

for any two functions φ and ω which verify the equation (17), the operator $\mathfrak{D}$ being given by (14).

Now, it is easy to see that the equation (18) leads immediately to the following formulae

$$\int_{\Gamma} \mathrm{M}\omega_1^* n_1 \, ds = \int_{\Gamma} x_1 \mathfrak{D}\omega_1^* \, ds ,$$

$$\int_{\Gamma} \mathrm{L}\omega_1^* n_2 \, ds = \int_{\Gamma} x_2 \mathfrak{D}\omega_1^* \, ds ,$$

$$\int_{\Gamma} \omega_1^* \mathfrak{D}\varphi \, ds = \int_{\Gamma} \varphi \mathrm{D}\omega_1^* \, ds ,$$

where φ satisfies the equation (17) and

$$\mathfrak{D}\varphi = \mathrm{M}x_2 n_1 - \mathrm{L}x_1 n_2 \qquad \text{on} \quad \Gamma .$$

The existence of the function φ is evident. The operator $\mathfrak{D}\omega_1^*$ can be easily calculated from (16) if we take into account (13).

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