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Local forms of invariant differential operators

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Matematica. — *Local forms of invariant differential operators.* Nota di ROBERT CARROLL (*), presentata (**) dal Socio Straniero A. WEINSTEIN.

RIASSUNTO. — Si dà una caratterizzazione dei coefficienti locali degli operatori invarianti differenziali e si stabiliscono alcune proprietà degli operatori stessi.

1\. Let G be a Lie group, $H \subset G$ a closed (Lie) subgroup, and write $h^\sim \subset g^\sim$ for the corresponding Lie algebras. Let V and W be finite dimensional (irreducible) complex H modules and write, for example, $V(G) = G \times_H V$ for the associated homogeneous vector bundle over $M = G/H$ (cfr. [9]); we recall that $V(G) = (G \times V)/H$ for the right H action $(g, v)h = (gh, h^{-1}v)$ on $G \times V$ (thus points of V(G) can be exhibited as $(g, v) \cdot H = \{(gh, h^{-1}v)\}$). Given a representation σ of H on V (defining V as an H module) one has an induced representation ρ of G on $C^\infty(V(G))$ by the rule $(\rho(g)\psi)(p) = g \cdot \psi(g^{-1}p)$ where $g \cdot (\hat{g}, v) \cdot H = (g\hat{g}, v) \cdot H$ and $\psi \in C^\infty(V(G))$ is a C^∞ section of V(G). A linear differential operator $D : C^\infty(V(G)) \to C^\infty(W(G))$ is called invariant if $D\rho(g) = \rho(g)D$ acting on sections ψ. We give a characterization of the local coefficients of such invariant operators and indicate some of their properties in various low rank situations. The details and further results will appear in [2, 3].

2\. Let $g^\sim = h^\sim + m^\sim$ be a vector space direct sum with X_i $(1 \leq i \leq r)$ a basis of $m^\sim$ and $X_{r+1}, \cdots, X_n$ a basis of $h^\sim$. Then one can choose local coordinates (gU, φ_g) on M where U is a suitable open neighborhood of $\theta = \pi(e)$ (π being the canonical map $G \to M$) and $\varphi_g : \pi(g \exp(\xi_1 X_1 + \cdots + \xi_r X_r)) \to (\xi_1, \cdots, \xi_r)$ (see [7] – recall also that the map $\xi \to \exp \sum_1^r \xi_i X_i \cdot \exp \sum_{r+1}^n \xi_i X_i$ will provide local coordinates on G in a neighborhood of e). One knows (cfr. [9]) that sections ψ of V(G) are in $1 - 1$ correspondence with maps $\hat{\psi} : G \to V$ satisfying $\hat{\psi}(gh) = h^{-1}\hat{\psi}(g)$ (the rule is determined by $\psi(gH) = (g, \hat{\psi}(g)) \cdot H$). Let $g_\xi = \exp(\xi_1 X_1 + \cdots + \xi_r X_r)$ and for $p = \pi(g) = \pi(g_\xi)$ define a trivialization of V(G) over U at p by the rule $T_p((g, v) \cdot H) = g_\xi^{-1} gv$ with inverse map $T_p^{-1}(v) = (g_\xi, v) \cdot H$; one translates this to any $\tilde{g}U$ in an obvious way. If $\mathfrak{I}$ denotes the corresponding trivialization in W(G) one writes $D = \mathfrak{I}^{-1}\tilde{D}T$ where $\tilde{D}$ is the local expression for D of the form $\tilde{D} = \Sigma\, a_\alpha(\xi)\, D^\alpha$ over U with $D^\alpha = D_1^{\alpha_1} \cdots D_r^{\alpha_r}$ and

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$D_k = \partial/\partial\xi_k$ ($a_\alpha(\xi) \in L(V,W)$). We will consider mainly first order operators D in this note. Now, corresponding to ρ, we have a representation $\hat{\rho}$ of G on functions $\hat{\psi}$ as above given by $(\hat{\rho}(g)\hat{\psi})(g^{\sim}) = \hat{\psi}(g^{-1}g^{\sim})$. Then writing out the invariance condition $\rho(g)D = D\rho(g)$ in local coordinates (U, φ_e) we obtain

LEMMA I. $\hat{\rho}(g)(\tilde{D}\hat{\psi})(g_\xi) = \tilde{D}\hat{\rho}(g)\hat{\psi}(g_\xi)$ *whenever* $g_\xi\theta$ *and* $g^{-1}g_\xi\theta$ *belong to* U.

Thus localized one easily establishes contact with the framework of [4, 12] where $G = P = T \times_\gamma L$ is the Poincaré group (cfr. also [8]); in fact the calculations of [4, 8, 12] have a version for any semidirect product $G = K \times_\gamma H$ and this is partially developed in [2]. Now we recall some algebraic constructions from [14]. Thus let $g_C^{\sim}$ denote the complexification of $g^{\sim}$ with $U_p(g_C^{\sim})$ the elements in the universal enveloping algebra $U(g_C^{\sim})$ of length not exceeding p and set $J_p(g_C^{\sim *}) = U_p(g_C^{\sim})^*$ (vector space dual). Define an action of $h_C^{\sim}$ on $U(g_C^{\sim}) \otimes V^*$ ($\otimes = \otimes_C$) by $r_{V^*}(Y)(u \otimes v') = r(Y)u \otimes v' + u \otimes Yv'$ where $r(Y)u = -uY$ and for $\alpha \otimes v \in J_p(g_C^{\sim *}) \otimes V$ one determines $r_V(Y)(\alpha \otimes v)$ by the rule $< r_V(Y)(\alpha \otimes v), u \otimes v' > = < \alpha \otimes v, r_{V^*}(Y^*)(u \otimes v') >$ where $Y^* = -Y$ and $u \otimes v' \in U_{p-1}(g_C^{\sim}) \otimes V^*$. The subspace of $J_p(g_C^{\sim *}) \otimes V$ annihilated by $r_V(Y)$ for $y \in h_C^{\sim}$ is denoted by $J_p(g_C^{\sim *}) \otimes^{h_C^{\sim}} V$ and is called a cotensor product (for $p = 0$ one identifies $J_0(g_C^{\sim *}) \otimes^{h_C^{\sim}} V$ with V). For $u \otimes v' \in U(g_C^{\sim}) \otimes V^*$ we write $\theta_{V^*}(Y)(u \otimes v') = \theta(Y)u \otimes v' + u \otimes Yv'$ where $\theta(Y)(X_1 \cdots X_p) = \Sigma X_1 \cdots [Y, X_i] \cdots X_p$ and then $J_p(g_C^{\sim *}) \otimes^{h_C^{\sim}} V$ becomes an $h_C^{\sim}$ module under the action of $\theta_V(Y)$ where $< \theta_V(Y)(\alpha \otimes v), u \otimes v' > = < \alpha \otimes v, \theta_{V^*}(Y^*)(u \otimes v') >$ for $u \otimes v' \in U_p(g_C^{\sim}) \otimes V^*$ (for $p = 0$, $\theta_V(Y)$ action coincides with Y action).

Now the jet bundle $J^p(V(G))$ is a homogeneous vector bundle $G \times_H Z$ where $Z = J_p(g_C^{\sim *}) \otimes^{h_C^{\sim}} V$ and invariant D of first order as above are associated with $h_C^{\sim}$ module homomorphisms $Q_1 : J_1(g_C^{\sim *}) \otimes^{h_C^{\sim}} V \to W$ (see [14]). We show that (under suitable identifications) $Q_1(j_{1,e}\hat{\psi}) = (\tilde{D}\hat{\psi})(e)$ and hence from lemma I it follows that $Q_1(j_{1,e}\hat{\psi}) = \Sigma a_\alpha(0) D^\alpha \hat{\psi}(g_\xi)|_{\xi=0}$. One now chooses $\hat{\psi} = f \otimes v$ with $f(e) = 0$ and f depending in our local coordinates on G only on ξ_m ($1 \le m \le r$) and there results

THEOREM I. *The local coefficients* $a_m(0)$ *of an invariant first order* D *are characterized by the rule* $Q_1(d\xi_m \otimes v) = a_m(0)v$.

Assume now that $h^{\sim}$ is reductive in $g^{\sim}$ so that $[Y, X_i] = \sum_{j=1}^{r} B_{ij}(Y) X_j$ for $Y \in h_C^{\sim}$ (this will occur for example if H is semisimple or compact or if $G = K \times_\gamma H$). Using the fact that Q_1 is an $h_C^{\sim}$ module homomorphism one proves

COROLLARY I. *For* $h^{\sim}$ *reductive in* $g^{\sim}$ *the characterization of theorem* I *yields the multiplication table*

$$[Y, a_m(0)] = -\sum_{i=1}^{r} B_{im}(Y) a_i(0). \tag{2.1}$$

We note that the invariance condition of lemma 1 for $\xi = 0$ describes local D action in terms of the $a_i(0)$ (cfr. also [7]). One can in fact derive (2.1) directly from this invariance statement in case $G = K \times_\gamma H$ for example (see [2]) and the calculations reduce to those of [8] when $G = P$; we also show that the $a_i(\xi)$ are constant when K is abelian. A characterization $Q_p(d\xi^\alpha \otimes v) = p!\, a_\alpha(0)\, v$ for $|\alpha| = p$ with Q_p an $\tilde{h}_C$ module homomorphism also holds for the top order coefficients $a_\alpha(0)$ when D is of order p and corollary 1 has an analogue for such a_α

$$(d\xi^\alpha = d\xi_{i_1}^{\alpha_{i_1}} \cdots d\xi_{i_n}^{\alpha_{i_n}} \text{ is the product in } J_p(\tilde{g}_C^{*}))\,.$$

3. Now (2.1) describes a representation of $\tilde{h}_C$ on the complex vector subspace $\mathcal{C}$ of $L(V, W)$ spanned by the $a_i(0)$. Thus, assuming $h^\sim$ is semisimple from now on, reduction of this representation involves classifying irreducible representations of $\tilde{h}_C = n^\sim$ on such $\mathcal{C}$ for $n^\sim$ either a simple complex Lie algebra $s^\sim$ or for $n^\sim = s^\sim \times s^\sim$ (cfr. [6]). Then let $c^\sim \subset n^\sim$ be a Cartan subalgebra with $n^\sim = c^\sim + \Sigma n^\alpha$ a root space decomposition (see [6, 7, 10, 13] for background here). If $\alpha_1, \cdots, \alpha_l$ is a base of roots with $m \geq l$ positive roots R_+ one finds elements $X_\alpha \in n^\alpha$, $Y_\alpha \in n^{-\alpha}$, and $H_\alpha \in c^\sim$ $(\alpha \in R_+)$ such that $n^\sim$ is generated by the X_{α_i}, Y_{α_i} and H_{α_i} $(1 \leq i \leq l)$ subject to the relations $[H_\alpha, H_\beta] = 0$, $[X_\alpha, Y_\alpha] = H_\alpha$, $[X_\alpha, Y_\beta] = 0$ if $\alpha \neq \beta$, $[H_\alpha, X_\beta] = n(\alpha, \beta) X_\beta$, $[H_\alpha, Y_\beta] = -n(\alpha, \beta) Y_\beta$, for $\alpha \neq \beta$, $\operatorname{ad}(X_\alpha)^{-n(\alpha,\beta)+1}(X_\beta) = 0$, and for $\alpha \neq \beta$ $\operatorname{ad}(Y_\alpha)^{-n(\alpha,\beta)+1}(Y_\beta) = 0$ where $\alpha, \beta \in R_+$, $n(\alpha, \beta) = 0, -1, -2$ or -3 for $\alpha \neq \beta$, and $n(\alpha, \alpha) = 2$. Irreducible representations of complex semisimple Lie algebras are characterized by a primitive element f of dominant weight $\tilde{w}$ (thus $X_\alpha f = 0$ and $Hf = \tilde{w}(H) f$ for $H \in c^\sim$) while elements $f_M = (\Pi Y_{\alpha_i}^{m_i}) f$, $1 \leq i \leq m$, will span the representation space F. Now choose suitable *bases* v'_M and w'_M for V and W from among the v_M and w_M indicated and try to find a suitable basis of matrices z'_M for $\mathcal{C}$ from among the z_M indicated, such that $[X_\alpha, z] = 0$ and $[H, z] = \tilde{w}(H) z$ for the primitive element z while certain "cutoff equations" indicated below are also satisfied. The z'_M will eventually be linear combinations of the $a_i(0)$ if an irreducible table (2.1) is given and, suitably ordered by i, will correspond (heuristically) to the local coefficients of $\partial/\partial\eta_i$ after a complex linear change of local coordinates $\xi \to \eta$. We will only concentrate on natural choices and properties of the z'_M although examples are worked out relating the z'_M to the $a_i(0)$.

First one looks at the lowest rank case $A_1 = sl(2, C) = so(3, C) = h^\sim$ (eventually we will treat all A_n, B_n, C_n, etc. in varying degrees of completeness—see [3]—but in this note only results for A_n will be indicated). Thus let $v_0 = v \in V$ (resp. $w_0 = w \in W$) be primitive and choose basis vectors $v_j = Y^j v$ (resp. $w_i = Y^i w$) where $0 \leq j \leq 2k$ (resp. $0 \leq i \leq 2k'$) with k (resp. k') an integer or half integer. Let there be $r = 2k'' + 1$ linear operators $z_m = [Y, z_{m-1}]$ $(z_0 = z)$ satisfying $[X, z] = 0$, $[H, z] = \tilde{w}(H) z = 2k'' z = (r-1) z$, and the "cutoff equation" $[Y, z_{r-1}] = 0$. We write

$z_i v_p = \Sigma \gamma^i_{p,j} w_j$ and determine the matrix entries $\gamma^i_{p,j}$ (p = row index and j = column index).

THEOREM 2. *$\gamma^0_{p,j} = 0$ unless $p - j = \beta + k - k'$ where $\beta = \left(\frac{r-1}{2}\right)$ and under this stipulation it is further necessary that the "cutoff condition"*

$$1 + \sum_{s=1}^{r} (-1)\binom{r}{s} \cdot$$

$$\cdot \frac{(p+s)\cdots(p+1)(2k-p)\cdots(2k-p-s+1)}{(k'-k+p-\beta+1)\cdots(k'-k+p-\beta+s)(k'+k-p+\beta-s+1)\cdots(k'+k-p+\beta)} = 0$$

holds. Under these circumstances one can prescribe arbitrarily the $\gamma^0_{p,j}$ of minimal j index, for example, satisfying $p - j = \beta + k - k'$ and then the remaining nonzero $\gamma^0_{p,j}$ are determined by the relation $\gamma^0_{p,j+1}(j+1)(2k'-j) = \gamma^0_{p-1,j}\, p\,(2k-p+1)$. The $\gamma^m_{p,j}$ for $1 \le m \le r-1$ are determined recursively from the equation $\gamma^m_{p,j} = \gamma^{m-1}_{p,j-1} - \gamma^{m-1}_{p+1,j}$ or directly from the formula $\gamma^m_{p,j} = \sum_{s=0}^{m} \binom{m}{s} (-1)^s \gamma^0_{p+s,j-m+s}$ and $\gamma^m_{p,j} = 0$ unless $p - j = \beta + k - k' - m$.

As an example, when $r = 3$ the cutoff condition holds when $k = k'$, $k = k' + 1$, or $k = k' - 1$. Then write, $\check{D}(\eta) = \sum_{1}^{3} z_i \eta_i$ and, taking $k = k'$, set $\Delta = \det \check{D}(\eta)$. In case $G = T \times_\gamma SO(3)$ is the natural semidirect product of the translations in R^3 with the rotation group one has $z_0 = -ia_1 + a_2$, $z_1 = 2ia_3$, and $z_2 = 2ia_1 + 2a_2$ (note that the matrices for the a_i will have complex entries). Making a suitable linear change of variables $\xi \to \zeta$, $\check{D}$ becomes $\check{D} = \sum_{0}^{2} z_i\, \partial/\partial\zeta_i$ and at the symbol level the variable change $y_1 = -i\eta_0 + 2i\eta_2$, $y_2 = \eta_0 + 2\eta_2$, and $y_3 = 2i\eta_1$ (y real) yields $\tilde{D}(y) = \Sigma a_i y_i = \check{D}(\eta)$. Using the fact that $-\eta_1^2 + 2\eta_0\eta_2 = \frac{1}{4}\Sigma y_i^2$ one can prove

THEOREM 3. *Let $r = 3$ and $k = k'$ with $G = T \times_\gamma SO(3)$. Then for k = integer, $\Delta = \det \tilde{D}(Y) = 0$ while for $k = (2n+1)/2$ $\tilde{D}$ is elliptic with $\Delta = c(\Sigma y_i^2)^{n+1}$ where $c = (2n-1)^2 \cdots 3^2/4^{n+1}(2n+1)^{2n}$ for $\left(n \ge 1 - c = \frac{1}{4} \text{ for } n = 0\right)$.*

4. Going next to $A_2 = sl(3, C)$ there are two basic roots α_1 and α_2 with $n(\alpha_1, \alpha_2) = n(\alpha_2, \alpha_1) = -1$ while $\alpha_1 + \alpha_2 = \alpha_3$ is also a root. If $\tilde{w}$ is the dominant weight for F with $\tilde{w}(H_1) = r$ and $\tilde{w}(H_2) = s$ ($H_i = H_{\alpha_i}$) then a basis for F can be expressed in the form $f_{i,j,k} = Y_3^i Y_2^j Y_1^k f$ where $0 \le k \le r$, $0 \le j \le s$, and $0 \le i \le r + s - j - k$ (see [1, 11]). Thus one seeks $z_{i'',j'',k''} = (ad\, Y_3)^{i''} (ad\, Y_2)^{j''} (ad\, Y_1)^{k''} z \in L(V, W)$ with $[X_1, z] = [X_2, z] = 0$, $[H_1, z] = r''z$, and $[H_2, z] = s''z$, while z must satisfy

the cutoff equations $(ad\ Y_1)^{r''+1} z = (ad\ Y_2)^{s''+1} z = 0$ (cfr. [10]). The representations of A_2 on V and W are supposed given here in terms of parameters (r, s) and (r', s') respectively. Then write $zv_{i,j,k} = \Sigma\, \gamma^{i,j,k}_{i',j',k'}\, w_{i',j',k'}$ and set $p = i + 2k - j$ with $q = i + 2j - k$; one uses indices p, q, i or i, j, k interchangeably.

THEOREM 4. *There are possible nonzero* $\gamma^{p,q,i}_{p',q',i'}$ *only when* $p - p' = R = r'' + r - r'$ *and* $q - q' = S = s'' + s - s'$. *These* γ *entries can then be computed from the equations* $[X_1, z] = [X_2, z] = 0$ *in the form*

$$(4.1) \qquad \begin{aligned} &(k'+1)(r'-k')\,\gamma^{p,q,i}_{p'+2,q'-1,i'} + (i'+1)\,\gamma^{p,q,i}_{p'+2,q'-1,i'+1} = \\ &= k(r-k+1)\,\gamma^{p-2,q+1,i}_{p',q',i'} + i\,\gamma^{p-2,q+1,i-1}_{p',q',i'}\,. \\ &\cdot\,(j'+1)(s'+k'-j'-i')\,\gamma^{p,q,i}_{p'-1,q'+2,i'} - (i'+1)\,\gamma^{p,q,i}_{p'-1,q'+2,i'+1} = \\ &= j(s+k-j-i+1)\,\gamma^{p+1,q-2,i}_{p',q',i'} - i\,\gamma^{p+1,q-2,i-1}_{p',q',i'} \end{aligned}$$

provided these equations have nontrivial solutions. Finally the cutoff conditions on the γ entries arising from the cutoff equations must be satisfied. The matrix entries for the remaining $z_{i'',j'',k''}$ can then be computed recursively from the definition of $z_{i'',j'',k''}$.

Various natural orderings of basis elements $v_{i,j,k}$ etc. can be found for any given nontrivial example in which symmetry properties etc. of the $z_{i'',j'',k''}$ are clearly exhibited; this will be developed in [3] and various patterns and qualitative features are visible. Now for $n > 2$ one does not have a table of basis vectors worked out in general for the irreducible representations of $A_n = sl(n+1, C)$ although presumably this could be done (after considerable calculation). Thus we have only studied examples involving fundamental weights $\tilde{w}_j$ (i.e., $\tilde{w}_j(H_k) = \delta_{j,k}$ with $1 \leq j, k \leq n = l$). If one writes $e_j(H) = \lambda_j$ for $H = \mathrm{diag}((\lambda_k)) \in c^{\sim}$ then

$$\tilde{w}_j = \left\{ (n+1-j) \sum_{k=1}^{j} e_k - j \sum_{k=j+1}^{n+1} e_k \right\} \Big/ (n+1)$$

and we obtain the other (equivalent and simple) weights in the representation determined by $\tilde{w}_j$ by the Weyl reflections corresponding to the independent permutations of the components e_k of $\tilde{w}_j$ (cfr. [5]). This enables one to readily determine natural basis vectors for $\tilde{w}_j$ and thence to obtain good matrices for the z'_M. Again various symmetry patterns are visible and properties of the z'_M will be discussed in [3].

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