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František Neuman

## Periodic Curvatures and Closed Curves

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**Geometria differenziale.** — *Periodic Curvatures and Closed Curves.*

Nota di FRANTIŠEK NEUMAN (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Richiamata la nozione generale di *curvatura* di una curva piana, si dimostra che – per una siffatta curva – l'essere *periodica* ogni sua curvatura non implica ancora la sua *chiusura*.

In this note it will be shown that the periodicity of any curvature from the class *K in any parametrization of the type *X of a closed plane curve does not imply that the curve is closed. *K and *X contain curvatures and parametrizations, respectively, introduced by Blaschke [1], Borůvka [2] and Santaló [5].

1. Let $u_1, u_2$ $(t \in \mathrm{I})$ be coordinate functions of a plane curve $\boldsymbol{u}$. Define an invariant curvature $k(t; \boldsymbol{u})$ of the curve $\boldsymbol{u}$ at a point $\boldsymbol{u}(t)$, $t \in \mathrm{I}$, as

$$(1) \qquad k(.\,; \boldsymbol{u}) : t \mapsto \mathrm{F}(u_1(t), u_2(t), u_1'(t), u_2'(t), \cdots, u_1^{(r)}(t), u_2^{(r)}(t))$$

for $t \in \mathrm{I}$, where $u_i^{(j)} = \mathrm{d}^j u_i / \mathrm{d}t^j$, $u_i^{(0)} = u_i$ for $j = 0, 1, \cdots, r$, $r \geqq 0$, $i = 1, 2$, and $\mathrm{F} : \mathrm{R}^{2(r+1)} \to \mathrm{R}$, R the set of all real numbers, such that $k(.\,; \boldsymbol{u})$ is invariant under any change of parameter $t$; i.e. $k(t_1; \boldsymbol{u}) = k(x_1, \boldsymbol{u}^*)$, where $x : t \mapsto x(t)$, $\mathrm{d}x(t)/\mathrm{d}t \neq 0$ for all $t \in \mathrm{I}$, $x_1 = x(t_1)$, $\boldsymbol{u}(u_1(t), u_2(t)) = \boldsymbol{u}^*(u_1^*(x), u_2^*(x))$, $u_i^*(x(t)) = u_i(t)$ for all $t \in \mathrm{I}$ and $i = 1, 2$. The existence of the necessary derivatives and the non-vanishing of denominators are assumed here and henceforth.

The definition includes, for example:

the curvature $k_1$ (unimodular affine invariant) in the sense of W. Blaschke [1, p. 13], which can be expressed (see [3]) as

$$k_1(.\,; \boldsymbol{u}) : t \mapsto$$

$$\mapsto \frac{1}{9}(-5[\boldsymbol{u}', \boldsymbol{u}''']^2 + 12[\boldsymbol{u}', \boldsymbol{u}''] \cdot [\boldsymbol{u}'', \boldsymbol{u}'''] + 3[\boldsymbol{u}', \boldsymbol{u}''] \cdot [\boldsymbol{u}', \boldsymbol{u}^{\mathrm{IV}}]) / [\boldsymbol{u}', \boldsymbol{u}'']^{8/3},$$

where

$$[\boldsymbol{u}^{(s)}, \boldsymbol{u}^{(p)}] = \begin{vmatrix} u_1^{(s)}(t) & u_1^{(p)}(t) \\ u_2^{(s)}(t) & u_2^{(p)}(t) \end{vmatrix};$$

the curvature $k_2$ (centroaffine invariant) in the sense of O. Borůvka [2, p. 29]:

$$k_2(.\,; \boldsymbol{u}) : t \mapsto \frac{\operatorname{sign}[\boldsymbol{u}, \boldsymbol{u}']}{2} \sqrt{\left|\frac{[\boldsymbol{u}, \boldsymbol{u}']}{[\boldsymbol{u}', \boldsymbol{u}'']}\right|} \cdot \left|3\,\frac{[\boldsymbol{u}, \boldsymbol{u}'']}{[\boldsymbol{u}, \boldsymbol{u}']} - \frac{[\boldsymbol{u}', \boldsymbol{u}''']}{[\boldsymbol{u}', \boldsymbol{u}'']}\right|.$$

(*) Mathematical Institute of the Academy, Department of Mathematics of J. E. P. University, Janáčkovo nám. 2a, Brno, Czechoslovakia. Research supported in part by NRC Grant A-2972.

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For necessary and sufficient condition on F to be a centroaffine (semi) invariant, see [3] or [4].

Let a parameter $x$,

$$x(\,.\,;t_0) : t \mapsto \int_{t_0}^{t} G\,(u_1^{(0)}(\sigma)\,,\,u_2^{(0)}(\sigma)\,,\cdots,\,u_1^{(r)}(\sigma)\,,\,u_2^{(r)}(\sigma))\,d\sigma\,, \tag{2}$$

$t_0 \in I\,, t \in I\,, r \geqq 0$, be introduced into our curve $\boldsymbol{u}$, where $G : R^{2(r+1)} \to R$, $x(\,.\,;t_0) : I \to J$, such that $x$ is invariant under any change of parameter, i.e. $x(t_1\,;t_0) = x(s_1\,;s_0)$ for all $t_1 \in I\,, t_0 \in I$, if $s : t \mapsto s(t)\,, ds(t)/dt \neq 0$ for all $t \in I$, and $s(t_1) = s_1\,, s(t_0) = t_0$.

In order that $x$ be a 1 — 1 mapping, let $G\,(u_1^{(0)}(t)\,,\cdots,\,u_2^{(r)}(t)) \neq 0$ for all $t \in I$. The last condition is a condition on G and/or $\boldsymbol{u}$.

Recall, for example, that

$$x_1 : t \mapsto \int_{t_0}^{t} [\boldsymbol{u}'(\sigma)\,,\,\boldsymbol{u}''(\sigma)]^{1/3}\,d\sigma$$

is the arc-length (unimodular affine invariant) in the sense of W. Blaschke [1, p. 8],

$$x_2 : t \mapsto \operatorname{sign}\,[\boldsymbol{u}\,,\boldsymbol{u}']\cdot\int_{t_0}^{t}\sqrt{\left|\frac{[\boldsymbol{u}'(\sigma)\,,\boldsymbol{u}''(\sigma)]}{[\boldsymbol{u}(\sigma)\,,\boldsymbol{u}'(\sigma)]}\right|}\,d\sigma$$

is the arc-length (centroaffine invariant up to the sign) in the sense of O. Borůvka [2, p. 28], and

$$x_3 : t \mapsto \frac{1}{2}\int_{t_0}^{t} \operatorname{sign}[\boldsymbol{u}(\sigma)\,,\boldsymbol{u}'(\sigma)]\cdot\frac{[\boldsymbol{u}'(\sigma)\,,\boldsymbol{u}''(\sigma)]}{[\boldsymbol{u}(\sigma)\,,\boldsymbol{u}'(\sigma)]^2}\,d\sigma\,,$$

or

$$x_3^* : t \mapsto \frac{1}{2}\int_{t_0}^{t} \operatorname{sign}[\boldsymbol{u}'(\sigma)\,,\boldsymbol{u}''(\sigma)]\cdot\frac{[\boldsymbol{u}'(\sigma)\,,\boldsymbol{u}''(\sigma)]}{[\boldsymbol{u}(\sigma)\,,\boldsymbol{u}'(\sigma)]^2}\,d\sigma$$

can be considered as the arc-length (unimodular centroaffine invariant) in the sense of L. A. Santaló (see [5, p. 96] and the discussion in [3]).

If a curve $\boldsymbol{u}$ is closed (not necessarily simply closed), then a parametrization $u_1 : p \mapsto u_1(p)\,,\; u_2 : p \mapsto u_2(p)\,,\; p \in (-\infty\,,\infty)$ always exists such that $u_i(p + c) = u_i(p)$ for a constant $c > 0$, every $p \in (-\infty\,,\infty)$ and $i = 1, 2$. Then $k(\,.\,;\boldsymbol{u}) : p \mapsto k(p\,;\boldsymbol{u})$ is periodic with period $c$ in the particular parametrization. If $x$ is an arbitrary parametrization given by (2), it can be expressed in terms of $p$ as

$$x(\,.\,;p_0) : p \mapsto \int_{p_0}^{p} G\,d\sigma\,.$$

From the periodicity of $u_i^{(j)}(p)$, we get

$$x(p+c\,;p_0)-x(p\,;p_0)=\int_p^{p+c}\mathrm{G}d\sigma=\mathrm{d}=\text{const.}\neq 0$$

for all $p\in(-\infty,\infty)$. Hence $x:(-\infty,\infty)\to(-\infty,\infty)$ is a surjection. Because $\mathrm{G}\neq 0$, the inverse $\pi$ of $x$ exists, $\pi:(-\infty,\infty)\to(-\infty,\infty)$ and

(3) $$\pi(\xi+\mathrm{d})=\pi(x(p)+\mathrm{d})=\pi(x(p+c))=p+c=\pi(\xi)+c$$

for $\xi=x(p)$ or $p=\pi(\xi)$, $\xi\in(-\infty,\infty)$. Since $k$ is invariant under any change of parameter, we have

$$k(p\,;\boldsymbol{u})\,|_{p=\pi(x)}=k(x\,;\boldsymbol{u}^*)\,,$$

where $\boldsymbol{u}^*(u_1^*(x)\,,u_2^*(x))$, $u_i^*(x(t))=u_i(t)$ for $i=1\,,2$ and $t\in(-\infty,\infty)$. However, for every $\xi\in(-\infty,\infty)$ and $p=\pi(\xi)$,

$$k(\xi+\mathrm{d}\,;\boldsymbol{u}^*)=k(x(p+c)\,;\boldsymbol{u}^*)=k(p+c\,;\boldsymbol{u})=$$
$$=k(p\,;\boldsymbol{u})=k(x(p)\,;\boldsymbol{u}^*)=k(\xi\,;\boldsymbol{u}^*).$$

Hence we have:

THEOREM 1. *If* $\boldsymbol{u}$ *is a closed curve, then any curvature* $k$ *satisfying* (1) *in any parametrization given by* (2) *is a periodic function.*

2. Consider a plane curve $\boldsymbol{u}$ given by $u_1:t\mapsto u_1(t)\in\mathrm{C}^n$, $u_2:t\mapsto u_2(t)\in\mathrm{C}^n$, where $\mathrm{C}^n$ $(n>0)$ denotes the set of all continuous functions on $(-\infty,\infty)$ having continuous derivatives up to and including the $n$–th oder. Moreover, let $[\boldsymbol{u}(t)\,,\boldsymbol{u}'(t)]\neq 0$ for all $t\in(-\infty,\infty)$. Let $^*\mathrm{C}^n$ denote the set of all such curves.

Consider all F in (1) and all G in (2) each depending only on $[\boldsymbol{u}^{(r)}\,,\boldsymbol{u}^{(s)}]$, $r\,,s=0\,,1\,,2\,,\cdots$ Denote the set of corresponding curvatures by *K and the set of corresponding parameters by *X.

It is obvious that $k_1\in{}^*\mathrm{K}$, $k_2\in{}^*\mathrm{K}$, $x_1\in{}^*\mathrm{X}$, $x_2\in{}^*\mathrm{X}$, $x_3\in{}^*\mathrm{X}$ and $x_3^*\in{}^*\mathrm{X}$.

Let us prove:

THEOREM 2. *If* $\boldsymbol{u}\in{}^*\mathrm{C}^1$ *is a closed curve, then any curvature* $k\in{}^*\mathrm{K}$ *in any parametrization* $x\in{}^*\mathrm{X}$ *is a periodic function. However, there exists an unbounded (hence not closed) curve* $\boldsymbol{u}\in{}^*\mathrm{C}^1$ *such that* $k(x\,;\boldsymbol{u})$ *is periodic for every* $k\in{}^*\mathrm{K}$ *and every* $x\in{}^*\mathrm{X}$.

*Proof*. The first part of the theorem is a special case of Theorem 1.

In order to construct an unbounded curve in $^*\mathrm{C}^2\subset{}^*\mathrm{C}^1$ having every curvature $k\in{}^*\mathrm{K}$ periodic in every parametrization $x\in{}^*\mathrm{X}$, let us consider the following parametrization of $\boldsymbol{u}\in{}^*\mathrm{C}^1$:

$$x_4:t\mapsto\int_{t_0}^{t}[\boldsymbol{u}(\sigma)\,,\boldsymbol{u}'(\sigma)]\,d\sigma\,.$$

Obviously $x_4 \in {}^*\mathrm{X}$ and

$$[\boldsymbol{u}(t), \boldsymbol{u}'(t)] = [\boldsymbol{u}^*(x_4), \mathrm{d}\boldsymbol{u}^*(x_4)/\mathrm{d}x_4] \cdot \mathrm{d}x_4(t)/\mathrm{d}t,$$

where $u_i^*(x_4)$ are the coordinate functions of $\boldsymbol{u}$ in the parametrization $x_4$, i.e. $u_i^*(x_4) = u_i(t)$ for $x_4 = x_4(t)$, for every $t \in (-\infty, \infty)$ and $i = 1, 2$. Also

$$\mathrm{d}x_4(t)/\mathrm{d}t = [\boldsymbol{u}(t), \boldsymbol{u}'(t)] \neq 0,$$

and we have

$$(4) \qquad [\boldsymbol{u}^*, \mathrm{d}\boldsymbol{u}^*/\mathrm{d}x_4] = 1.$$

If, in addition, $\boldsymbol{u} \in {}^*\mathrm{C}^2$, then $[\boldsymbol{u}^*, \mathrm{d}^2\boldsymbol{u}^*/\mathrm{d}x_4^2] = 0$, that is $u_i^*(x_4), i = 1, 2$, are linearly independent solutions of a differential equation

$$(q) \qquad \mathrm{d}^2 y/\mathrm{d}x_4^2 = q(x_4)\, y,$$

where

$$(5) \qquad q(x_4) = [\mathrm{d}^2\boldsymbol{u}^*/\mathrm{d}x_4^2, \mathrm{d}\boldsymbol{u}^*/\mathrm{d}x_4] \in \mathrm{C}^0.$$

Conversely, any pair of linearly independent solutions of $(q)$ can be considered as a pair of coordinate functions of a curve $\boldsymbol{u}$ belonging to ${}^*\mathrm{C}^2$. If the function $q$ in $(q)$ is periodic, then any curvature $k \in {}^*\mathrm{K}$ of $\boldsymbol{u}$ in the parametrization $x_4$ must be periodic, since, according to (4) and (5), every determinant

$$[\mathrm{d}^r\boldsymbol{u}^*/\mathrm{d}x_4^r, \mathrm{d}^s\boldsymbol{u}^*/\mathrm{d}x_4^s]$$

can be written as a function of $q$ and its derivatives. Then, however, $k \in \mathrm{K}^*$ is periodic in any parametrization $x \in {}^*\mathrm{X}$, since G is also periodic (with period $c$) in $x_4$ and $x \in {}^*\mathrm{X}$ must satisfy

$$x(p + c; p_0) - x(p; p_0) = \int_p^{p+c} \mathrm{G}\,\mathrm{d}x_4 = \mathrm{d} = \text{const.} \neq 0,$$

or

$$k(\xi + \mathrm{d}, \boldsymbol{u}^{**}) = k(p + c, \boldsymbol{u}^*) = k(p, \boldsymbol{u}^*) = k(\xi, \boldsymbol{u}^{**}), \ \xi = x(p),$$

for $k \in \mathrm{K}^*$ and $x \in {}^*\mathrm{X}$, where $\boldsymbol{u}^{**}$ is the parametrization of $\boldsymbol{u}$ in $x$.

From the Floquet Theory it follows that there exist unbounded solutions of differential equations $(q)$, $q \in \mathrm{C}^0$, $q$ periodic. Hence the proof of Theorem 2 is complete.

The following corollary is a direct consequence of Theorem 2. The existence of the necessary derivatives and the non-vanishing of denominators are assumed.

COROLLARY. *If $\boldsymbol{u}$ is a closed plane curve, then both Blaschke's curvature $k_1$ and Borůvka's curvature $k_2$ of $\boldsymbol{u}$ are periodic functions in Blaschke's parametrization $x_1$, Borůvka's parametrization $x_2$ and Santalo's parametrization $x_3$ (or $x_3^*$). However, neither the periodicity of $k_1$ nor the periodicity of $k_2$ of $\boldsymbol{u}$ in any of the parametrizations $x_1, x_2, x_3$ and $x_3^*$ is sufficient for $\boldsymbol{u}$ to be closed.*

This is a generalisation of some results in [3] or [4].

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