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A note on commutators

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Analisi funzionale. — *A note on commutators*. Nota di MENDEL DAVID, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota viene dimostrato il seguente teorema: Siano A, B e C operatori lineari definiti in uno spazio di Hilbert tale che AB — BA = C. Se il contorno del campo numerico di A contiene un punto dello *sp* A allora inf $|(Cx, x)| = 0$.

1. INTRODUCTION. Let H be a Hilbert Space and suppose that A, B and C are (bounded linear) operators on H such that

$$AB - BA = C. \tag{1}$$

C. R. Putnam proved that if A is semi-normal then

$$\inf_{\|x\|=1} |(Cx, x)| = 0. \tag{2}$$

On the other hand, (2) is not necessarily satisfied if A is not semi-normal (see [3] p. 8, 9). However the methods used in [1] and a lemma proved in [4] will allow us to prove the following

THEOREM. *Let* A, B *and* C *be operators satisfying* (1). *If the boundary of the numerical range of* A *contains a point of sp* A *then* (2) *holds.*

Remark. It is known that a semi-normal operator is spectraloid and convexoid (for definitions see [2] pp. 114–115) while there exist spectraloid operators that are not convexoid (see [2], pp. 322–323). Since the conclusion of the above Theorem is true, in particular, when A is spectraloid it follows that the Theorem is a generalization of Putnam's result.

2. In the proof of the Theorem the following known Lemma will be used.

LEMMA. *If* E *is an operator such that*

$$\operatorname{Re}(Ex, x) \geq 0, \qquad x \in H$$

then

$$\|E^* x\| \leq (6 \|E\| \, \|Ex\| \, \|x\|)^{1/2}, \qquad x \in H$$

(see [4], p. 251).

Proof of the Theorem. Let α be a spectral point of A on the boundary of the numerical range. Since the numerical range of A is convex there exists a complex number θ with $|\theta| = 1$ such that $D = \theta(A - \alpha)$ satisfies

$$\operatorname{Re}(D^* x, x) = \operatorname{Re}(Dx, x) \geq 0, \qquad x \in H \tag{3}$$

(*) Nella seduta del 9 maggio 1970.

Let $\{a_n\}$ be a sequence of negative numbers converging to 0. Let $D_n = D - a_n$ for $n = 1, 2, \cdots$ From (3) we have

$$\mathrm{Re}\,(D_n^* x, x) \geq 0, \qquad x \in H$$

Therefore, by the Lemma,

$$\| D_n x \| \leq (6 \| D_n \| \, \| D_n^* x \| \, \| x \|)^{1/2} \tag{4}$$

for all x in H.

An easy computation shows that (1) implies

$$D_n B - B D_n = C'$$

where $C' = \theta C$. It follows that

$$B (D_n^*)^{-1} - D_n^{-1} B D_n (D_n^*)^{-1} = D_n^{-1} C' (D_n^*)^{-1}. \tag{5}$$

Let y be a vector in H with $\| y \| = 1$. By (5) we have

$$\| D_n^{-1} C' (D_n^*)^{-1} y \| \leq \| B (D_n^*)^{-1} y \| + \| D_n^{-1} B D_n (D_n^*)^{-1} y \| \leq \tag{6}$$
$$\leq \| B \| \, \| D_n^{-1} \| \, (1 + \| D_n (D_n^*)^{-1} y \|).$$

Now, putting $x = (D_n^*)^{-1} y$ in (4) one obtains

$$\| D_n (D_n^*)^{-1} y \| \leq (6 \| D_n \| \, \| (D_n^*)^{-1} y \|)^{1/2} \leq (6 \| D_n \| \, \| D_n^{-1} \|)^{1/2}.$$

Thus (6) implies

$$\| D_n^{-1} C' (D_n^*)^{-1} \| \leq \| B \| \, \| D_n^{-1} \| \, (1 + (6 \| D_n \| \, \| D_n^{-1} \|)^{1/2}). \tag{7}$$

On the other hand one has

$$\| D_n^{-1} C' (D_n^*)^{-1} y \| \geq | (D_n^{-1} C' (D_n^*)^{-1} y, y) | = | (C' (D_n^*)^{-1} y, (D_n^*)^{-1} y) | =$$
$$= \frac{| (C' (D_n^*)^{-1} y, (D_n^*)^{-1} y) |}{\| (D_n^*)^{-1} y \|^2} \, \| (D_n^*)^{-1} y \|^2.$$

Therefore

$$\| D_n^{-1} C' (D_n^*)^{-1} y \| \geq \inf \frac{| (C' x, x) |}{\| x \|^2} \, \| (D_n^*)^{-1} y \|^2. \tag{8}$$

Since y is an arbitrary vector with $\| y \| = 1$ it follows from (8) that

$$\| D_n^{-1} C' (D_n^*)^{-1} \| \geq \inf \frac{| (C' x, x) |}{\| x \|^2} \, \| D_n^{-1} \|^2 = \inf_{\| x \| = 1} | (C' x, x) | \, \| D_n^{-1} \|^2.$$

Hence, by (7)

$$\inf_{\| x \| = 1} | (Cx, x) | = \inf_{\| x \| = 1} | (C' x, x) | \leq \| B \| \left(\frac{1}{\| D_n^{-1} \|} + \left(\frac{6 \| D_n \|}{\| D_n^{-1} \|} \right)^{1/2} \right) \tag{9}$$

Since $0 \in sp\,D$, $-\frac{1}{a_n} \in sp\,D_n^{-1}$. Hence $(\|D_n^{-1}\|)^{-1} \leq |a_n|$. It follows from (9) that

$$\inf_{\|x\|=1} |(Cx, x)| \leq \|B\| (|a_n| + (6\|D_n\| \, |a_n|)^{1/2}). \tag{10}$$

Since $a_n \to 0$ and $\|D_n\| \to \|D\|$ as $n \to \infty$ (2) follows from (10). The Theorem is proved.

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[4] T. I. SEIDMAN, *An identity for normal-like operators*, « Israel J. Math. », 7, 249–253 (1969).