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Equazioni differenziali. — *Note on a certain non-autonomous differential equation.* Nota di ROLF REISSIG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si prosegue lo studio, iniziato in [1], di una classe di equazioni differenziali ordinarie (1). Si danno condizioni sufficienti per l'esistenza delle soluzioni periodiche nel caso in cui la $f(x)$ è non lineare e non limitata.

In a foregoing paper [1] we considered the differential equation

$$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f(x) = p(t) \tag{1}$$
$$[n \geqq 1 \quad , \quad p(t+\omega) \equiv p(t)]$$

where the functions f and p are continuous for all values x respectively t and the positive constant coefficients a_i $(1 \leqq i \leqq n)$ fulfil the Hurwitz conditions for the n-th order algebraic equation

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 .$$

Following to the Leray-Schauder fixed point technique we proved:

THEOREM 1. *Equation* (1) *admits at least one periodic solution of period* ω *if*

(i) $|f(x)| \leqq F$ *for all* x

(ii) $f(x) \operatorname{sgn} x > 0$ *for* $|x| \geq h > 0$

(iii) $|p(t)| \leqq m \quad , \quad |P(t)| = \left| \int_0^t p(\tau)\, d\tau \right| \leqq M .$

For each periodic solution $x(t)$ we derived a bound of the type

$$A = h + k(m + M + F) \geqq S = \operatorname*{Max}_{[0,\omega]} |x(t)| \tag{2}$$

where the parameter k is only depending on the coefficients $a_1, \cdots, a_n$.

Now it is easy to show that an ω-periodic solution is existing too if boundedness condition (i) is replaced by the weaker one

$$\limsup_{|x| \to \infty} \left| \frac{f(x)}{x} \right| < \frac{1}{k} \tag{i'}$$

(Theorem 1').

A consequence of this condition is an estimate

$$\left| \frac{f(x)}{x} \right| \leqq \frac{\vartheta}{k} \qquad (|x| \geqq X) \tag{3}$$

with an adequate number $\vartheta \in (0, 1)$.

(*) Nella seduta del 9 maggio 1970.

Let F be an arbitrarily chosen positive number such that

$$F \geqq \frac{1}{\vartheta} \max_{|x| \leqq X} |f(x)| ,$$

$$F \geqq \frac{\vartheta}{1-\vartheta} \frac{h+k(m+M)}{k} ;$$

be

$$A = h + k(m + M + F) .$$

In the case

$$\max_{|x| \leqq A} |f(x)| > F$$

we conclude:

$$\frac{1}{A} \max_{|x| \leqq A} |f(x)| > \frac{F}{A} = \frac{F}{h+k(m+M+F)}$$

$$= \frac{1}{k + \frac{h+k(m+M)}{F}}$$

$$\geqq \frac{1}{k + \frac{1-\vartheta}{\vartheta} k}$$

$$= \frac{\vartheta}{k} ,$$

$$\frac{1}{A} \max_{|x| \leqq X} |f(x)| \leqq \frac{\vartheta F}{h+k(m+M+F)} < \frac{\vartheta}{k}$$

and therefore

$$X < A , \qquad \max_{X \leqq |x| \leqq A} \left| \frac{f(x)}{x} \right| \geqq \frac{1}{A} \max_{X \leqq |x| \leqq A} |f(x)|$$

$$= \frac{1}{A} \max_{|x| \leqq A} |f(x)| > \frac{\vartheta}{k}$$

on the contrary to (3). Consequently we obtain

(4) $$|f(x)| \leqq F \quad \text{for} \quad |x| \leqq A .$$

Instead of differential equation (1) the nonlinear term of which satisfies (i'), (ii) we consider another one,

(1*) $$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f^*(x) = p(t) ,$$

where

$$f^*(x) = \begin{cases} f(x) & ; \quad |x| \leqq A \\ f(A \operatorname{sgn} x) & ; \quad |x| \geqq A . \end{cases}$$

We see immediately that in the case of equation (1*) all assumptions of Theorem 1 are fulfilled; for this reason at least one ω-periodic solution

$x(t)$ is ensured. Since this solution can be estimated according to (2) it is also a solution of the original equation (1).

Remark. The proof of Theorem 1′ cannot be carried out by a similar straightforward application of Sedziwy's boundedness theorem based on conditions (i)–(iii), see [2].

Bibliography.

[1] R. Reissig, *On the existence of periodic solutions of a certain non-autonomous differential equation.* «Ann. Mat. Pura Appl.», (4), *85*, 235–240 (1970).

[2] S. Sedziwy, *Asymptotic properties of solutions of nonlinear differential equations of the higher order.* Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Nr. CXXXI, 69–80 (1966).