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Derivative and continuity in a linear topological space

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 9 maggio 1970

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Derivative and continuity in a linear topological space.* Nota di NANIGOPAL BISWAS, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Vi si definisce la derivata di una funzione che trasforma in sè medesimo uno spazio topologico lineare in maniera che, se lo spazio è, in particolare, l'asse reale, si perviene alla nozione elementare di derivata.

In recent years some authors (see for example [1], [2]) have attempted to introduce the definition of the derivative of a function in a linear topological space X. It appears that none of them have shown the connection between their definitions of derivatives and the ordinary definition of derivatives when X is the real number space with the natural topology. The main purpose of the present paper is to show that it is possible to define the derivative of a function mapping a linear topological space into itself in such a way that the definition coincides with the ordinary definition of derivative when X is the real number space with the natural topology. Our approach is axiomatic and the conditions used here may be compared with the corresponding conditions in [3] where F. W. Perkins has characterized by a set of conditions those real functions of the real variable x which are everywhere differentiable. Also we have obtained certain conditions which characterize the set of all continuous functions $f: X \to X$.

Let X be a linear topological space over a scalar field Φ of real numbers. By a linear topological space we mean an abstract space with a topology in which the functions $x + y$ and λx where $x, y \in X$ and $\lambda \in \Phi$ are continuous functions of both the variables.

(*) Nella seduta dell'11 aprile 1970.

SECTION I. We consider a set D of functions $f: X \to X$. We suppose that to every function f belonging to D there corresponds a unique function $\bar{f}: X \to X$ (not necessarily belonging to D). We suppose that the set D of functions and the correspondence between f and $\bar{f}$ satisfy the following conditions.

(1) (a) The identity function $I(x) = x$ is in D.
(b) For any element $x_1 \in X$, $\bar{I}(x_1)$ is a fixed non-null element of X which we denote by α.

(2) Given any element $x_2 \in X$ and any function $f \in D$, then $F(x) = f(x + x_2)$ is a function in D and $\bar{F}(x) = \bar{f}(x + x_2)$.

(3) There exists an element $x_3 \in X$ such that if f_1 and f_2 are functions in D and k an arbitrary element of Φ, then the function $F(x) = f_1(x) + kf_2(x)$ is a function in D and $\bar{F}(x_3) = \bar{f}_1(x_3) + k\bar{f}_2(x_3)$.

(4) There exists a constant x_4 such that if f_1 and f_2 belong to D and if $f_1(x_4) = f_2(x_4) = \bar{f}_1(x_4) = \theta$ and $\bar{f}_2(x_4) \neq \theta$, then corresponding to every such pair of functions and any neighbourhood Δ of $f_1(x_4)$ there exists a neighbourhood δ of x_4 such that

$$f_1(\bar{\delta}_1) \subset \{f_2(\bar{\delta}_1)\} \cup \{-f_2(\bar{\delta}_1)\} \subset \Delta$$

for any neighbourhood δ_1 where $x_4 \in \delta_1 \subset \delta$.

PROPOSITION 1. If $f_1(x)$ and $f_2(x)$ be any two functions in D and if λ and μ be any two elements of Φ, then $G(x) = \lambda f_1(x) + \mu f_2(x)$ is a function in D and $\bar{G}(x) = \lambda \bar{f}_1(x) + \mu \bar{f}_2(x)$.

Proof. Replacing $f_1(x)$ and $f_2(x)$ by $I(x)$ and putting $K = -1$ in condition 3, we get that the function $\chi(x) = \theta$ belongs to D and $\bar{\chi}(x_3) = \theta$. Putting $f_1(x) = \chi(x)$ in condition 3 we get that $F(x) = \chi(x) + \mu f_2(x) = \mu f_2(x)$ belongs to D and $\bar{F}(x_3) = \mu \bar{f}_2(x_3)$. Similarly it follows that $\Psi(x) = \lambda f_1(x)$ belongs to D and $\bar{\Psi}(x_3) = \lambda \bar{f}_1(x_3)$. Thus $G(x) = \lambda f_1(x) + \mu f_2(x)$ belongs to D and $\bar{G}(x_3) = \lambda \bar{f}_1(x_3) + \mu \bar{f}_2(x_3)$. We now prove that for any arbitrary element ξ of X, $\bar{G}(\xi) = \lambda \bar{f}_1(\xi) + \mu \bar{f}_2(\xi)$. By condition 2, the functions $g_1(x) = f_1(x + \xi - x_3)$, $g_2(x) = f_2(x + \xi - x_3)$ and $g_3(x) = G(x + \xi - x_3)$ (where ξ is any element of X) belong to D. So, by our previous considerations, the functions $\lambda g_1(x)$, $\mu g_2(x)$ and consequently $g_3(x) = \lambda g_1(x) + \mu g_2(x)$ belong to D. By condition 3, we get $\bar{g}_3(x_3) = \lambda \bar{g}_1(x_3) + \mu \bar{g}_2(x_3)$. This implies however that $\bar{G}(\xi) = \lambda \bar{f}_1(\xi) + \mu \bar{f}_2(\xi)$.

PROPOSITION 2. If $A(x) = ax + b\alpha$ where a and b are any two elements of Φ, then $A(x)$ belongs to D and $\bar{A}(x) = a\alpha$.

Proof. Consider the function $B(x) = b\alpha$. We write $B(x) = I(x + b\alpha) - I(x)$. Therefore by conditions 1, 2 and 3 $B(x)$ is in D. Since for any element $x_2 \in X$, $B(x + x_2) = B(x)$ and so $\bar{B}(x + x_2) = \bar{B}(x)$, $\bar{B}(x)$ is a constant function. In particular, the function $\mathfrak{A}(x) = \alpha$ is in D and $\bar{\mathfrak{A}}(x) =$ a constant which we denote by $\overline{\mathfrak{A}}$. We have $B(x) = b\alpha = b\mathfrak{A}(x)$ and so $\overline{B}(x) = b\overline{\mathfrak{A}}$. Now $I(x+\alpha) = x+\alpha = I(x) + \mathfrak{A}(x)$. For any element $x_1 \in X$ $\bar{I}(x_1+\alpha) = \bar{I}(x_1) + \overline{\mathfrak{A}}(x_1)$, which by condition 1 (*b*) becomes $\alpha = \alpha + \overline{\mathfrak{A}}(x_1)$. Therefore $\overline{\mathfrak{A}}(x_1) = \theta$ for any x_1. Since $A(x) = ax + b\alpha = aI(x) + b\mathfrak{A}(x)$, we have $\bar{A}(x_1) = a\bar{I}(x_1) + b\overline{\mathfrak{A}}(x_1) = a\alpha$ for any $x_1 \in X$.

COROLLARY. *Any constant function $f(x) = \beta$, where $\beta \in X$ belongs to* D *and $\bar{f}(x) = \theta$ for any $x \in X$.*

Proof. We have $f(x) = \beta = I(x+\beta) - I(x)$. By conditions 2 and 3 $f(x)$ belongs to D and $\bar{f}(x) = \bar{I}(x+\beta) - \bar{I}(x) = \alpha - \alpha = \theta$.

PROPOSITION 3. Let $f_1(x)$ and $f_2(x)$ belong to D and let ξ be any element of X such that $f_1(\xi) = f_2(\xi) = \bar{f}_1(\xi) = \theta$ and $\bar{f}_2(\xi) \neq \theta$. Then corresponding to every neighbourhood Δ of $f_1(\xi)$ there exists a neighbourhood δ of ξ such that

$$f_1(\bar{\delta}_1) \subset \{f_2(\bar{\delta}_1)\} \cup \{-f_2(\bar{\delta}_1)\} \subset \Delta$$

for any neighbourhood δ_1 where $\xi \in \delta_1 \subset \delta$.

Proof. Let $g_1(x) = f_1(x - x_4 + \xi)$ and $g_2(x) = f_2(x - x_4 + \xi)$ where x_4 is the constant of condition 4. By condition 2 we have $g_1(x)$ and $g_2(x)$ belong to D and $\bar{g}_1(x) = \bar{f}_1(x - x_4 + \xi)$ and $\bar{g}_2(x) = \bar{f}_2(x - x_4 + \xi)$. Now $g_1(x_4) = g_2(x_4) = \bar{g}_1(x_4) = \theta$, $\bar{g}_2(x_4) \neq \theta$. Hence by the condition 4, corresponding to a neighbourhood Δ of $g_1(x_4)$ there exists a neighbourhood δ' of x_4 such that

$$g_1(\bar{\delta}_1') \subset \{g_2(\bar{\delta}_1')\} \cup \{-g_2(\bar{\delta}_1')\} \subset \Delta \quad \text{for} \quad x_4 \in \delta_1' \subset \delta'.$$

This is equivalent to

$$f_1(\bar{\delta}_1) \subset \{f_2(\bar{\delta}_1)\} \cup \{-f_2(\bar{\delta}_1)\} \subset \Delta \quad \text{for} \quad \xi \in \delta_1 \in \delta.$$

DEFINITION. Let $f(x)$ be a function mapping X into itself. We say that $f(x)$ is differentiable in X if $f(x)$ belongs to some set D for which the conditions 1–4 hold and in this case the associated function $\bar{f}(x)$ is said to be the derivative of $f(x)$.

When X is the real number space with the natural topology, the justification of the definition of the derivative follows from the following theorem.

THEOREM 1. *Let* X *be the real number space with the natural topology and let $\alpha = 1$. Then a necessary and sufficient condition that a function $f(x)$ has a derivative for every value of x is that $f(x)$ belongs to some set* D *for which conditions* 1–4 *hold. If $f(x)$ belongs to any such set* D, *then the associated function $\bar{f}(x)$ is necessarily the derivative of $f(x)$.*

Proof. If D′ is the set of all real functions $f(x)$ which are everywhere differentiable and $f'(x) = \bar{f}(x)$, then clearly the conditions 1, 2, 3 are satisfied. That condition 4 is also satisfied can be seen as follows. Let $f_1(x)$ and $f_2(x)$ be two functions such that $f_1(x_4) = f_2(x_4) = f_1'(x_4) = 0$ and $\bar{f}_2(x_4) \neq 0$, then it is clear that there exists a neighbourhood δ, $|x - x_4| \leq \varepsilon$ say, such that $f_1(x) \leq |f_2(x)|$ for x in δ. Replacing $f_1(x)$ by $-f_1(x)$ we have for x in δ, $-f_1(x) \leq |f_2(x)|$. Therefore $-|f_2(x)| \leq f_1(x) \leq |f_2(x)|$ i.e. $|f_1(x)| \leq |f_2(x)|$. This shows that $f_1(\overline{\delta}) \subset \{f_2(\overline{\delta})\} \cup \{-f_2(\overline{\delta})\}$. The result is clearly true for any neighbourhood δ_1 such that $x_4 \in \delta_1 \subset \delta$. Since $f_1(x)$ and $f_2(x)$ are continuous functions, corresponding to a neighbourhood δ of 0 $(= f_1(x_4))$ the neighbourhood δ can be so adjusted that the condition 4 holds.

We now show that if $f(x) \in \mathrm{D}$, then $f(x)$ is differentiable everywhere and $f'(\xi) = \overline{f}(\xi)$ for arbitrary ξ. Let $f_2(x) = \lambda x - \lambda\xi$ where λ is any positive constant and let $f_1(x) = f(x) - \{\overline{f}(\xi)\, x + [f(\xi) - \xi\overline{f}(\xi)]\}$. Then $f_1(x)$ is in D and $\overline{f}_1(x) = \overline{f}(x) - \overline{f}(\xi)$. Since $f_1(\xi) = f_2(\xi) = \overline{f}_1(\xi) = 0$ and $\overline{f}_2(\xi) = \lambda \neq 0$, we have by proposition 3, corresponding to any neighbourhood Δ of $f_1(\xi)$ there exists a neighbourhood $\delta = (\xi - \eta, \xi + \eta)$ of ξ such that

$$f_1(\overline{\delta}_1) \subset \{f_2(\overline{\delta}_1)\} \cup \{-f_2(\overline{\delta}_1)\} \subset \Delta \quad \text{where} \quad \xi \in \delta_1 \subset \delta .$$

Therefore for all x in $[\xi - \eta', \xi + \eta']$, $0 < \eta' \leq \eta$, we have

$$\lambda(\xi - \eta') - \lambda\xi \leq f_1(x) \leq \lambda(\xi + \eta') - \lambda\xi \quad \text{or,} \quad -\lambda\eta' \leq f_1(x) \leq \lambda\eta'$$

$$\therefore \quad -\lambda\eta' \leq f_1(\xi + \eta') - f_1(\xi) \leq \lambda\eta' \qquad (\text{since } f_1(\xi) = 0)$$

$$\therefore \quad \left| \frac{f_1(\xi + \eta') - f_1(\xi)}{\eta'} \right| \leq \lambda .$$

Since $\lambda > 0$ is arbitrary, we have $f_1'(\xi) = 0$. This implies that $f'(\xi)$ exists and $f'(\xi) = \overline{f}(\xi)$.

SECTION II. Let C be the set of functions $f : \mathrm{X} \to \mathrm{X}$ such that

(i) for any $\lambda \in \mathrm{X}$ the constant function $f(x) = \lambda$ is in C.

(ii) if f_1 and f_2 belong to C and if $f_1(\xi) = f_2(\xi)$ then for any neighbourhood Δ of $f_1(\xi)$ there exists a neighbourhood δ of ξ such that $f_1(\delta), f_2(\delta) \subset \Delta$.

THEOREM 2. *A necessary and sufficient condition that* $f : \mathrm{X} \to \mathrm{X}$ *be continuous is that* $f(x)$ *belongs to some set* C *for which the conditions* i) *and* ii) *hold.*

Proof. It is clear that if C′ is the set of all functions which are continuous then the conditions i) and ii) are satisfied.

Conversely, let us suppose that $f(x)$ is an arbitrary function in C and let $f(\xi) = \lambda$. Let us consider the constant function $f_1(x) = \lambda$. Then f and f_1 belong to C and $f(\xi) = f_1(\xi)$. Thus corresponding to any neighbourhood Δ of $f(\xi)$ there exists a neighbourhood δ of ξ such that $f(\delta), f_1(\delta) \subset \Delta$. This shows that $f(x)$ is continuous at ξ. Since ξ is an arbitrary point, the theorem follows.

Theorem 3. *If a function* $f: X \rightarrow X$ *is differentiable, then it is continuous.*

Proof. Since $f(x)$ is differentiable, $f(x) \in D$. It follows from the discussions on the conditions and with the help of proposition 3 that $D \subset C$. Consequently $f(x)$ is continuous.

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References.

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