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A further result on the approximation of set-valued mappings

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Analisi funzionale. — *A further result on the approximation of set-valued mappings.* Nota di ARRIGO CELLINA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Il risultato principale del lavoro è il seguente: siano X e Y spazi metrici localmente convessi e Γ una multiapplicazione semicontinua superiormente da X ai convessi totalmente limitati di Y. Allora, dato $\varepsilon > 0$, esiste una applicazione univoca continua $f: X \rightarrow Y$ tale che la distanza di Hausdorff tra il grafico di f e quello di Γ è minore di ε.

INTRODUCTION.

This Note is concerned with the approximation of upper semi-continuous set-valued mappings by means of continuous single-valued mappings. Results on this problem have been presented in [1] and [2] where it was shown that, under certain conditions, it is possible to find a continuous single-valued mapping whose graph is "close" to the graph of the set-valued mapping, i.e. such that the separation of the graph of the single-valued mapping to the graph of the set-valued mapping is small. It is our present purpose to prove a stronger approximation result, i.e. to show that it is possible to find continuous single-valued approximations such that the Hausdorff distance of the graphs of the approximating functions to the graph of the set-valued mapping can be made arbitrarily small. Although this result in a certain sense gives a better approximation than the results presented in [1] and [2], all these results are independent of each other. In fact, for instance, while in the above mentioned papers the approximating mappings were found to be finite dimensional, this property is not essential for the present purpose.

NOTATIONS AND BASIC DEFINITIONS.

If S is a metric space, x and $s \in S$, then $d(x, s)$ denotes the distance of x from s. If Z is also a metric space, $S \times Z$ is a metric space with $d((s, z), (x, y)) = \max\{d(s, x), d(z, y)\}$. For $A \subset S$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. An open *ball* about x of radius $\varepsilon > 0$ is denoted by $B[x, \varepsilon]$. We also set $B[A, \varepsilon] = \{y \in S : d(y, A) < \varepsilon\}$ and $\operatorname{diam}(A) = \sup\{d(x, y) : x \text{ and } y \in A\}$ when the supremum exists. For A and B contained in S we define the *separation* of A from B, $d^*(A, B)$, to be $\sup\{d(a, B) : a \in A\}$ when the supremum exists. For those A and B such that both $d^*(A, B)$ and $d^*(B, A)$ are defined, we set the *Hausdorff distance* between A and B to be $\delta(A, B) = \sup\{d^*(A, B), d^*(B, A)\}$. We remark that, although the word distance has been used, δ will not satisfy the requirements of a metric, since in particular we do not limit ourselves to closed subsets of S.

(*) Nella seduta dell'11 aprile 1970.

When Y is a linear space, K (Y) will denote the set of *convex* subsets of Y. A mapping $\Gamma : S \to K(Y)$ will be considered as a set-valued mapping. By its graph we mean the subset of $S \times Y$ defined by

$$G = \{(s, y) : s \in S \quad \text{and} \quad y \in \Gamma(s)\}$$

A mapping $\Gamma : S \to K(Y)$ is called *upper semi-continuous* (u.s.c.) at s if $\Gamma(s) \neq \varnothing$ and if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Gamma(B[s, \delta]) \subset \subset B[\Gamma(s), \varepsilon]$, Γ is called *u.s.c.* on S if it is *u.s.c.* at each point $s \in S$.

RESULTS.

We shall need the following

PROPOSITION 1. Let X be a topological space whose non-empty open sets have infinite cardinality. Let $U : I \to 2^X$ be an injection from an index set I, such that for all $i \in I$, $U(i)$ is non-empty and open and such that the family $\{U(i) : i \in I\}$ is neighbourhood finite in X. Then there exists an injection $\beta : I \to X$ that is a selection from U, that is such that $\beta(i) \in U(i)$, $\forall i \in I$.

Proof. Let $J \subseteq I$ and $\alpha : J \to X$ be an injection such that $\alpha(i) \in U(i)$, $\forall i \in J$. Let B be the set of all such injections

$$B = \{\alpha \mid \alpha : J \to X ; \; J \subseteq I ; \quad \alpha \text{ injective}; \quad \alpha(i) \in U(i) \quad \forall i \in I\}$$

It is clear that B is non-empty. Partially order B by setting

$$\alpha_1 \leq \alpha_2 \Longleftrightarrow \{D(\alpha_1) \subseteq D(\alpha_2) \quad \text{and} \quad \alpha_2 |_{D(\alpha_1)} = \alpha_1\}$$

where $D(\alpha)$ is the domain of α.

It is straightforward to prove that each totally ordered set admits an upper bound. Therefore, by Zorn's Lemma, there exists a maximal element $\beta \in B$. We claim that $D(\beta) = I$. Suppose there exists an $i_0 \in I - D(\beta)$. Let $V \subset U_{i_0}$ be an open neighbourhood of x_{i_0} meeting a finite number of U_i. Since for all $i \in D(\beta)$ it is $\beta(i) \in U_i$, V contains a finite number of $\beta(i)$. Let $x \in V$ be such that $x \neq \beta(i)$, $i \in D(\beta)$. Setting $\beta'(i) = \beta(i)$ for all $i \in D(\beta)$ and $\beta'(i_0) = x$, we have that $\beta' \in B$ and that $\beta \leq \beta'$ while it is not true that $\beta' \leq \beta$. This contradicts the maximality of β. Therefore $D(\beta) = I$.

In what follows we shall consider metric locally convex spaces (l.c.s.). We shall assume that the metric has been chosen such that balls are convex. The following Lemma will be used:

LEMMA 1. *Let* K *be a convex subset of the metric l.c.s.* X, *such that diam* (K) $< \varepsilon/2$, *and* K *contains at least two points. Then given a totally bounded convex subset T of the metric l.c.s.* Y, *there exists a continuous single-valued function* $f : K \to T$ *such that*

$$\delta(F, K \times T) < \varepsilon$$

where F *is the graph of* f.

Proof. When T consists of a single point, the Lemma is trivial. Assume that T consists of more than one point, and suppose first that T is contained in a finite dimensional subspace $Y' \subset Y$. In Y' let us choose a cube C containing T and let r be a retraction mapping C onto $\bar{T}$ (such a retraction exists by a theorem of Dugundji).

Since K contains two points, it contains their convex combination I. Let φ be a spacefilling curve from I onto C [4]: then $r(\varphi(x))$ is a spacefilling curve from I onto $\bar{T}$. Let r_1 be a retraction mapping K onto I and set $f(x) = r(\varphi(r_1(x)))$. Since the range of f is in $\bar{T}$, $d^*(F, K\times T) = 0$. Let $(x, y) \in K \times T$. Then there exists an $x' \in I$ such that $y = r(\varphi(x')) = r(\varphi(r_1(x)))$ and

$$d((x, y), F) \leq d((x, y), (x', y)) + d((x', y), F) \leq \varepsilon/2 < \varepsilon.$$

Now let T be not finite dimensional. Since it is totally bounded, there exists a finite number of points y_i, $i = 1, 2, \cdots, N$, such that $\bigcup_{i=1}^{N} B[y_i, \varepsilon/2]$ covers T. Let $T' = T \cap$ span $\{y_i\}$ and let f be defined as above, such that $\delta(F, K \times T') \leq \varepsilon/2$. Then again $d^*(F, K\times T) = 0$ and for a given $(x, y) \in K \times T$, there exists a $y_i \in T'$ such that $d((x, y), F) \leq d((x, y), (x, y_i)) + d((x, y_i), F) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $d^*(F, K\times T) < \varepsilon$, concluding the proof.

The following is our main result:

THEOREM. *Let* X *and* Y *be metric l.c.s., let* $\Gamma : X \to K(Y)$ *be u.s.c. and such that for all* $x \in X$, $\Gamma(x)$ *is totally bounded. Then given* $\varepsilon > 0$, *there exists a continuous single-valued mapping* $f : X \to Y$ *such that*

$$\delta(F, G) < \varepsilon$$

where F *and* G *are the graphs of* f *and* Γ.

Proof. For each $x \in X$, let $\delta = \delta(x)$, $0 < \delta(x) < \varepsilon/4$ be such that $\Gamma(B[x, \delta(x)]) \subset B[\Gamma(x), \varepsilon/2]$. Then $\cup B[x, \delta(x)/2]$ covers X. Since X is paracompact, let $\{U_i, i \in I\}$ be a neighbourhood finite open refinement of $\{B[x, \delta(x)/2]\}$. We can assume that the mapping $i \to U_i$ is injective. Then, by Proposition 1, there exists an injective selection $i \to x_i$. To each x_i we can associate an η'_i such that $\overline{B[x_i, \eta'_i]} \subset U_i$ and $\overline{B[x_i, \eta'_i]}$ meets only a finite number of U_i, say U_{i_k}, $k = 1, 2, \cdots, n_i$. Set $\eta''_i = \min\{d(x_i, x_{i_k}) : k = 1, \cdots, n_i\}$ and $\eta_i = 1/3 \min\{\eta'_i, \eta''_i\}$. Consider the family $\{\overline{B[x_i, \eta_i]}\}_{i\in I}$. We claim that $\overline{B[x_i, \eta_i]} \cap \overline{B[x_j, \eta_j]} = \varnothing$ for i and $j \in I$, $i \neq j$. Suppose this is not true for $i = i^0$, $j = j^0$ and that $\eta_{i^0} \geq \eta_{j^0}$. Then $x_{j^0} \in V_{i^0}$ and therefore $d(x_{i0}, x_{j0}) \geq 3\eta_{i0} \geq 3\eta_{j0}$, that contradicts the assumption.

For each $i \in I$ consider the set

$$C_i = \bigcup_{\substack{j\in I \\ j \neq i}} \overline{B[x_j, \eta_j]}.$$

Since the family $\{\overline{B[x_j, \eta_j]}\}_{j\in I}$ is neighbourhood finite, C_i is closed [3].

The sets $O_i = U_i - C_i$ are non-empty (since $\overline{B[x_i, \eta_i]} \subset O_i$) and open. Moreover $\{O_i\}_{i \in I}$ is a neighbourhood finite covering of X with the property that each O_i contains a non-empty closed and convex set, $\overline{B[x_i, \eta_i]}$, such that $\overline{B[x_i, \eta_i]} \cap O_j = \varnothing$ for $i \neq j$. Let $\{p_i(\cdot)\}_{i \in I}$ be a partition of unity subordinated to $\{O_i\}$. Since $\{U_i\}$ is a refinement of $\{B[x, \delta(x)/2]\}$, to each $i \in I$ let us associate a ξ_i such that $O_i \subset U_i \subset B[\xi_i, \frac{1}{2}\delta_i]$, where we set $\delta_i = \delta(\xi_i)$. Let $\gamma_i : \overline{B[x_i, \eta_i]} \to \Gamma(\xi_i)$ be a continuous function such that

$$\delta(\text{graph}(\gamma_i), \overline{B[x_i, \eta_i]} \times \Gamma(\xi_i)) < \varepsilon .$$

Such a function exists by the preceding Lemma. Let f_i be an extension of γ_i to O_i with values in $\Gamma(\xi_i)$. Such an extension exists by a theorem of Dugundji, since $\Gamma(\xi_i)$ is convex. Set

$$f(x) = \sum_{i \in I} p_i(x) f_i(x) .$$

This is the required function. It is known that f is well defined and continuous. Let $x \in X$ and i_k, $k = 1, 2, \cdots, n$ be such that $p_{i_k} \neq 0$. Let k^0 be such that $\delta_{i_{k^0}} \geq \delta_{i_k}$, $k = 1, \cdots, n$. Then $f(x)$ is a convex combination of points $f_{i_k}(x) \in \Gamma(\xi_{i_k})$. By construction all the ξ_{i_k} are contained in $B[\xi_{i_{k^0}}, \delta_{i_{k^0}}]$ and by the definition of $\delta_{i_{k^0}}$, $f_{i_k}(x) \in B[\Gamma(\xi_{i_{k^0}}), \varepsilon/2]$. But then, also

$$f(x) \in B[\Gamma(\xi_{i_{k^0}}), \varepsilon/2].$$

It follows that

$$d((x, f(x)), G) \leq d((x, f(x)), (\xi_{i_{k^0}}, f(x))) + d((\xi_{i_{k^0}}, f(x)), G) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

i.e.

$$d^*(F, G) < \varepsilon .$$

Now let $(x, y) \in G$. Then x belongs to some $O_i \subset B[\xi_i, \delta_i]$ and there exists a $y_i \in B[y, \varepsilon/2] \cap \Gamma(\xi_i)$. By construction, there exists an $x' \in B[x_i, \eta_i]$ such that $d((x_i, y_i), (x', \gamma_i(x'))) < \varepsilon/2$. Since by construction

$$f|_{B[x_i, \eta_i]} = \gamma_i ,$$

it is

$$d((x, y), (x', f(x'))) \leq d((x, y), (x_i, y_i)) + d((x_i, y_i), (x', f(x'))) <$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

i.e.

$$d^*(G, F) < \varepsilon$$

concluding the proof.

Remark. Γ need not be defined on the whole X. The theorem holds for Γ defined on any non-empty closed and convex subset of X, with at least two points.

References.

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[4] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.