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On a class of integro-differential equations. Nota IV

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Analisi matematica. — *On a class of integro-differential equations.* Nota IV di Mehmet Namik Oğuztöreli (*) e Demetrio Mangeron (**) (***) (1), presentata (****) dal Socio M. Picone.

Riassunto. — Si studia un problema al contorno per un'equazione integro-differenziale alle derivate parziali del secondo ordine incontratasi in un problema di controllo ottimale considerato in [1].

1. Introduction.

In the present article we consider the following integro-differential equation

$$(1.1) \qquad \lambda \frac{\partial^2 u(x,y)}{\partial x^2} - u(x,y) = f(x,y) + \mu \iint_R K(x,y;\xi,\eta)\, u(\xi,\eta)\, d\xi\, d\eta$$

for $(x,y) \in R$, subject to the boundary conditions

$$(1.2) \qquad u(0,y) = u(1,y) = 0 \qquad \text{for} \quad 0 \leq y \leq 1,$$

where $R = \{(x,y) \mid 0 \leq x, y \leq 1\}$, λ and μ are real parameters, $f(x,y)$ and $K(x,y;\xi,\eta)$ are given functions which are continuous on R and $R \times R$ respectively, and $u(x,y)$ is the unknown function. In the following we investigate continuous solutions of the boundary value problem (1.1)-(1.2). This problem occurs in an optimization problem involving a distributed parameter control system considered in [1].

2. Solution for small μ.

In this section we seek a solution of the form

$$(2.1) \qquad u(x,y) = \sum_{n=0}^{\infty} \mu^n u_n(x,y) \quad , \quad u_n(0,y) = u_n(1,y) = 0$$

for the boundary value problem (1.1)-(1.2). The following recursive relation-

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ships can be easily established:

$$(2.2)\quad \begin{cases} \lambda \dfrac{\partial^2 u_0(x,y)}{\partial x^2} - u_0(x,y) = f(x,y), \\ \lambda \dfrac{\partial^2 u_n(x,y)}{\partial x^2} - u_n(x,y) = \displaystyle\iint_R K(x,y;\xi,\eta)\, u_{n-1}(\xi,\eta)\, d\xi\, d\eta, \\ \qquad\qquad n = 1, 2, 3, \cdots \end{cases}$$

or, equivalently,

$$(2.3)\quad u_n(x,y) = -\frac{1}{\lambda}\varphi_n(x,y) - \frac{1}{\lambda}\int_0^1 G(x,\sigma)\, u_n(\sigma,y)\, d\sigma, \qquad n = 0, 1, 2, \cdots$$

where

$$(2.4)\quad \begin{cases} \varphi_0(x,y) = \displaystyle\int_0^1 G(x,\sigma) f(\sigma,y)\, d\sigma, \\ \varphi_n(x,y) = \displaystyle\iint_R H(x,y;\xi,\eta)\, u_{n-1}(\xi,\eta)\, d\xi\, d\eta, \qquad n = 1, 2, 3, \cdots \end{cases}$$

and

$$(2.5)\quad \begin{cases} H(x,y;\xi,\eta) = \displaystyle\int_0^1 G(x,\sigma) K(\sigma,y;\xi,\eta)\, d\sigma \\ G(x,\sigma) = \dfrac{2}{\pi^2}\displaystyle\sum_{k=1}^{\infty} \dfrac{\sin k\pi x \sin k\pi\sigma}{k^2} = \begin{cases} \sigma(1-x) & 0 \le \sigma \le x \le 1, \\ x(1-\sigma) & 0 \le x \le \sigma \le 1. \end{cases} \end{cases}$$

It is well-known that the eigenvalues and eigenfunctions of the symmetric kernel $G(x,\sigma)$ are $\lambda_k = k^2\pi^2$ and $\theta_k(x) = \sqrt{2}\sin k\pi x$, $k = 1, 2, 3, \cdots$, respectively. Thus, if $\lambda \neq -\frac{1}{k^2\pi^2}$, then for each n, Eq. (2.3) admits a unique solution which is given by the formula

$$(2.6)\quad \begin{cases} u_0(x,y) = -\dfrac{1}{\lambda}\varphi_0(x,y) + \dfrac{1}{\lambda^2}\displaystyle\int_0^1 \Gamma\left(x,\sigma;\frac{-1}{\lambda}\right)\varphi_0(\sigma,y)\, d\sigma \\ u_n(x,y) = -\dfrac{1}{\lambda}\displaystyle\iint_R \Omega\left(x,y;\xi,\eta;\frac{-1}{\lambda}\right) u_{n-1}(\xi,\eta)\, d\xi\, d\eta, \quad n = 1, 2, 3, \cdots \end{cases}$$

where

$$(2.7)\quad \Omega(x,y;\xi,\eta;\nu) = H(x,y;\xi,\eta) + \nu\int_0^1 \Gamma(x,\sigma;\nu)\, H(\sigma,y;\xi,\eta)\, d\sigma$$

with $\nu = -\frac{1}{\lambda}$. Writing $\|w\| = \max_{R} |w(x,y)|$ for any $w(x,y)$ defined on R, and putting

$$\begin{cases} K = \left\| \iint_R |K(x,y;\xi,\eta)| \, d\xi \, d\eta \right\| \quad , \quad H = \left\| \iint_R |H(x,y;\xi,\eta)| \, d\xi \, d\eta \right\| \\ \Gamma_\nu = \|\Gamma(x,\sigma;\nu)\| \quad , \quad \nu = -\frac{1}{\lambda}, \end{cases} \tag{2.8}$$

and remembering that $H \leq \frac{K}{8}$, we easily show that

$$\begin{cases} \|u_n\| \leq \left(\frac{K}{8}\right)^n (1 + |\nu| \, \Gamma_\nu)^n \, \|u_0\| , \\ \left\| \frac{\partial^2 u_n}{\partial x^2} \right\| \leq \left(\frac{K}{8}\right)^n (9 + |\nu| \, \Gamma_\nu)^n \, [\|u_0\| + \|f\|] . \end{cases} \tag{2.9}$$

Hence by the M–test of Weierstrass, both of the series $\sum_{n=0}^{\infty} \mu^n u_n(x,y)$ and $\sum_{n=0}^{\infty} \mu^n \frac{\partial^2 u_n(x,y)}{\partial x^2}$ are absolutely and uniformly convergent on the square R if

$$|\mu| < \frac{8}{(9 + |\nu| \, \Gamma_\nu) K}, \tag{2.10}$$

where $\nu = -\frac{1}{\lambda}$ (2). Thus, we have

THEOREM 1. *For* $\lambda \neq -\frac{1}{k^2 \pi^2}$, $k = 1, 2, 3, \cdots$, *and for* μ *satisfying the inequality* (2.10) *with* $\nu = -\frac{1}{\lambda}$, *the boundary value problem* (1.1)–(1.2) *admits a unique continuous solution* $u(x, y)$ *which is given by the formulas* (2.1) *and* (2.6).

3. Solution for large λ and arbitrary μ.

In this section we establish a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} \lambda^{-n} u_n(x,y) , \qquad u_n(0,y) = u_n(1,y) = 0 \tag{3.1}$$

for the boundary value problem (1.1)–(1.2). We can easily show that the functions $u_n(x,y)$ satisfy the following equations:

$$\begin{cases} \frac{\partial^2 u_1(x,y)}{\partial x^2} = f(x,y) \\ \frac{\partial^2 u_n(x,y)}{\partial x^2} = u_{n-1}(x,y) + \mu \iint_R K(x,y;\xi,\eta) \, u_{n-1}(\xi,\eta) \, d\xi \, d\eta , \\ \hfill n = 2, 3, 4, \cdots . \end{cases} \tag{3.2}$$

(2) Tha same conclusions are valid for the first derivatives series.

Thus, we have

$$\text{(3.3)}\quad \begin{cases} u_1(x,y) = -\int\limits_0^1 G(x,\sigma) f(\sigma,y)\, d\sigma , \\ u_n(x,y) = -\int\limits_0^1 G(x,\sigma) u_{n-1}(\sigma,y)\, d\sigma - \mu \iint\limits_R H(x,y;\xi,\eta)\, u_{n-1}(\xi,\eta)\, d\xi\, d\eta, \\ \qquad n = 2, 3, 4, \cdots . \end{cases}$$

As in § 2, we can easily establish the following inequalities:

$$\text{(3.4)}\quad \begin{cases} \| u_n \| < \frac{1}{8} \left(\frac{1 + |\mu| K}{8} \right)^{n-1} \| f \| \\ \left\| \frac{\partial^2 u_n}{\partial x^2} \right\| < \left(\frac{1 + |\mu| K}{8} \right)^{n-1} \| f \| \end{cases} \qquad n = 1, 2, 3, \cdots .$$

Thus, the series $\sum_{n=1}^{\infty} \lambda^{-n} u_n(x,y)$ and $\sum_{n=1}^{\infty} \lambda^{-n} \frac{\partial^2 u_n(x,y)}{\partial x^2}$ are absolutely and uniformly convergent on the square R for λ satisfying the inequality

$$\text{(3.5)}\quad |\lambda| > \frac{8}{1 + |\mu| K}$$

where μ is arbitrary (3). Hence, we may state the following

THEOREM 2. *The boundary value problem* (1.1)–(1.2) *admits a unique continuous solution for* λ *satisfying the inequality* (3.5) *and for arbitrary* μ, *and this solution is given by the formulas* (3.1) *and* (3.3).

BIBLIOGRAPHY.

[1] M. N. OĞUZTÖRELI, *On the Optimization in Distributed Parameter Control Systems* (in press).

[2] M. N. OĞUZTÖRELI and D. MANGERON, *On a Class of Integro-Differential Equations. V* (in press).

[3] M. PICONE – T. VIOLA, *Lezioni Sulla Teoria Moderna dell'Integrazione*, Edizioni Scientifiche Einaudi, Roma 1952.

[4] V. I. SMIRNOV, *Integral Equations and Partial Differential Equations*, Volume IV. *A Course of Higher Mathematics*. Pergamon Press, London 1964.

(3) One obtains similarly the same conclusions concerning the first derivatives series.