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The Unitary Brauer-Rademacher Identity

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Teoria dei numeri. — *The Unitary Brauer–Rademacher Identity.* Nota di LESLIE E. SHADER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — L'identità di Brauer–Rademacher viene stabilita con metodo più semplice dell'usuale, nell'ipotesi che ci si limiti a divisori « unitari » nel senso qui più oltre precisato. Ulteriori estensioni vengono analogamente ottenute sopra GF$[q, x]$.

The Brauer–Rademacher identity

$$\varphi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\mu(r/d)/\varphi(d) = \mu(r) \sum_{d \mid (r,n)} d\mu(r/d) \tag{1}$$

has been subjected to many proofs and generalizations [1], [5], and [6]. A further immediate generalization is to GF $[q, x]$, the ring of polynomials in one variable over the finite field GF(q). In which case (1) holds with integers replaced by polynomials in the argument of the various functions the function, d replaced by $|\mathrm{D}| = q^{\deg \mathrm{D}}$, and all divisors restricted to being primary.

It is the purpose of this note to generalize the Brauer–Rademacher identity by restricting all divisors to be " unitary ". That is, if d/r and $(d, r/d) = 1$, then d is called a *unitary divisor of r*. Thus we have the Unitary Brauer–Rademacher identity given in

THEOREM I.

$$\varphi^*(r) \sum_{\substack{d \| r \\ (n,d)_*=1}} d\mu^*(r/d)/\varphi^*(d) = \mu^*(r) \sum_{d \| (n,r)_*} d\mu^*(r/d), \tag{2}$$

where φ^*, μ^*, $(a, b)_*$ *are the unitary analogs of the usual* φ, μ, *and* (a, b) *functions* [2].

Note: It is of some interest that the methods employed in the proof of (1) in [1], [5], and [6], are not applicable here. However, the proof of Theorem I is elementary, requiring only the property that both members of (2) are multiplicative in r.

Proof: Since $c^*(n, r) = \sum_{d \| (n,r)_*} d\mu^*(r/d)$, where $c^*(n, r)$ is the unitary Ramanujan sum, and both $c^*(n, r)$ and $\mu^*(r)$ are multiplicative in r, the right member, $\mathrm{R}(r)$, of (2) is multiplicative in r. Let $r = r_1 r_2$, $(r_1, r_2) = 1$. Then $d \| r$ implies that $d = d_1 d_2$ where $d_1 \| r_1$, $d_2 \| r_2$ and $(d_1, d_2)_*$. Also $(n, d)_* = (n, d_1)_* (n, d_2)_*$, and φ^* is multiplicative in r.

(*) Nella seduta dell'11 aprile 1970.

Thus, if $L(r)$ is the left member of (2),

$$L(r) = \varphi^*(r_1)\,\varphi^*(r_2) \sum_{\substack{d_1 \| r_1 \\ (n,d_1)_*=1}} \; \sum_{\substack{d_2 \| r_2 \\ (n,d_2)_*=1}} d_1 d_2\, \mu^*(r_1 r_2/d_1 d_2)/\varphi^*(d_1 d_2)$$

$$= \varphi^*(r_1) \sum_{\substack{d_1 | r_1 \\ (n,d_1)_*=1}} d_1\, \mu^*(r_1/d_1)/\varphi^*(d_1)\; \varphi^*(r_2) \sum_{\substack{d_2 | r_2 \\ (n,d_2)_*=1}} d_2\, \mu^*(r_2/d_2)/\varphi^*(d_2)\,,$$

and $L(r)$ is multiplicative in r. Thus we need only consider (2) with $r = p^K$, p prime and K a positive integer.

But

$$L \equiv L(p^K) = (p^K - 1) \begin{cases} -1 + p^K/(p^K - 1)\,, & \text{if } p^K \nmid n \\ -1\,, & \text{if } p^K | n\,. \end{cases}$$

$$= \begin{cases} 1\,, & \text{if } p^K \nmid n \\ 1 - p^K, & \text{if } p^K | n\,. \end{cases}$$

Similarly,

$$R = R(p^K) = (-1) \begin{Bmatrix} -1\,, & \text{if } p^K \nmid n \\ -1 + p^K, & \text{if } p^K | n \end{Bmatrix} = L\,,$$

which establishes the theorem.

Theorem I can, of course, be generalized to $GF[q, x]$ by replacing integers with polynomials, replacing the function d by $|D| = q^{\deg D}$ and replacing the number theoretic functions φ^*, μ^* $(a, b)_*$ by their $GF[q, x]$ analogs, (see [3]).

References.

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[5] Subbarao M. V., *The Brauer–Rademacher Identity*, «American Mathematical Monthly», *72*, 135–138.

[6] Szusz P., *Once More the Brauer–Rademacher Identity*, «American Mathematical Monthly», *74*, 570–571 (1967).