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On characterizations of semi—continuous functions

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'11 aprile 1970

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On characterizations of semi-continuous functions.*
Nota di NANIGOPAL BISWAS, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Data la nozione di funzione semicontinua in un insieme semiaperto di uno spazio topologico arbitrario, vi si danno criteri sufficienti affinché tale semicontinuità sussista.

The following definitions are known [3].

DEFINITION. A set A in a topological space X will be said to be semi-open (written s.o.) if and only if there exists an open set O such that $O \subset A \subset \overline{O}$ ($\overline{O}$ denotes the closure of O in X).

DEFINITION. Let $f: X \to X^*$ be single-valued (continuity not assumed) where X and X^* are topological spaces. Then $f: X \to X^*$ is termed semi-continuous (written s.c.) if and only if, for O^* open in X^*, then $f^{-1}(O^*)$ is s.o. in X.

The purpose of the present note is to explore certain characterizations of semi-continuous functions, in the light of the characterizations of continuous functions as given in [2] (page 86). For this, we introduce first some definitions and discuss some of their immediate consequences.

DEFINITION 1. A set A in a topological space X is said to be semi-closed if and only if there exists a closed set F such that $\operatorname{Int} F \subset A \subset F$.

(*) Nella seduta dell'11 aprile 1970.

THEOREM 1. *A set* A *in* X *is semi-closed if and only if* $\text{Int}\,\bar{A} \subset A$.

Proof. Suppose first that A is semi-closed. There exists a closed set F such that $\text{Int}\, F \subset A \subset F$. This implies that $\text{Int}\,\bar{A} \subset \text{Int}\, F \subset A$.

Next suppose that $\text{Int}\,\bar{A} \subset A$. Putting $\bar{A} = F$, we get $\text{Int}\, F \subset A \subset \bar{A} = F$. This shows that A is semi-closed.

THEOREM 2. *A set* A *in* X *is semi-closed if and only if* cA *is semi-open* (cA *denotes the complement of* A *in* X).

Proof. Let A be semi-closed. Then there exists a closed set F such that $\text{Int}\, F \subset A \subset F$. This implies that $cF \subset cA \subset c\, \text{Int}\, F = \overline{cF}$. Consequently cA is semi-open.

Conversely, suppose that cA is semi-open. Then there exists an open set O such that $O \subset cA \subset \bar{O}$, i.e. $c\bar{O} \subset A \subset cO$. Thus $\text{Int}\, cO = c\bar{O} \subset A \subset cO$. Putting $cO = F$, we get $\text{Int}\, F \subset A \subset F$ which shows that A is semi-closed.

THEOREM 3. *The intersection of an arbitrary collection of semi-closed sets is semi-closed.*

The proof is immediate. It may be shown by examples that the union of two semi-closed sets is not necessarily semi-closed.

Remark. A closed set is necessarily semi-closed. The converse, however, is false.

DEFINITION 2. The intersection of all semi-closed sets containing a set A is said to be the semi-closure of A and is denoted by $C_s(A)$.

Remark. The set $C_s(A)$ is semi-closed.

DEFINITION 3 (cf. [1]). A set $M_x \subset X$ is said to be a semi-neighbourhood of a point $x \in X$, if and only if there exists a semi-open set A such that $x \in A \subset M_x$.

THEOREM 4. *Let* $f: X \to X^*$ *be single-valued where* X *and* X^* *are topological spaces. Then the following statements are equivalent.*

(a) The function f is semi-continuous.

(b) The inverse of each open set is semi-open.

(c) For each point p in X and each open set O^* in X^* with $f(p) \in O^*$, there is a semi-open set A in X such that $p \in A$, $f(A) \subset O^*$.

(d) The inverse of each closed set is semi-closed.

(e) For each x in X, the inverse of every neighbourhood of $f(x)$ is a semi-neighbourhood of x.

(f) For each x in X and each neighbourhood $\mathfrak{U}$ of $f(x)$, there is a semi-neighbourhood V of x such that $f(V) \subset \mathfrak{U}$.

(g) For each subset A of X, $f[C_s(A)] \subset \overline{f(A)}$.

(h) For each subset B of X^*, $C_s[f^{-1}(B)] \subset f^{-1}(\bar{B})$.

Proof:

(a) $\longleftrightarrow$ (b): This follows from the definition of semi-continuous functions.

(a) $\longleftrightarrow$ (c): This follows from Theorem 12 [3].

(b) $\longleftrightarrow$ (d): The result follows from the fact that if B is a subset of X^*, then $f^{-1}(cB) = cf^{-1}(B)$.

(b) $\rightarrow$ (e): For x in X, let V be a neighbourhood of $f(x)$. Then $f(x) \in O^* \subset V$ where O^* is open in X^*. Consequently, $f^{-1}(O^*)$ is a semi-open set in X and $x \in f^{-1}(O^*) \subset f^{-1}(V)$. Thus $f^{-1}(V)$ is a semi-neighbourhood of x.

(e) $\rightarrow$ (f): Let $x \in X$ and $\mathfrak{A}$ be a neighbourhood of $f(x)$. Then $V = f^{-1}(\mathfrak{A})$ is a semi-neighbourhood of x and $f(V) = f\{f^{-1}(\mathfrak{A})\} \subset \mathfrak{A}$.

(f) $\rightarrow$ (c): For x in X, let O^* be an open set containing $f(x)$. Then O^* is a neighbourhood of $f(x)$. So, there exists a semi-neighbourhood V of x such that $x \in V$ and $f(V) \subset O^*$. Hence there exists a semi-open set A in X such that $x \in A \subset V$. Consequently, $f(A) \subset f(V) \subset O^*$.

(g) $\longleftrightarrow$ (d): Suppose that (d) holds and let A be a subset of X. Since $A \subset f^{-1}f(A)$, we have $A \subset f^{-1}\{\overline{f(A)}\}$. Now $\overline{f(A)}$ is a closed set in X^* and hence $f^{-1}\{\overline{f(A)}\}$ is a semi-closed set containing A. Consequently, $C_s(A) \subset f^{-1}\{\overline{f(A)}\}$.

$$\therefore \quad f\{C_s(A)\} \subset ff^{-1}\{\overline{f(A)}\} \subset \overline{f(A)}.$$

Conversely, suppose that (g) holds for any subset A of X. Let F be a closed subset of X^*. Then $f\{C_s[f^{-1}(F)]\} \subset \overline{ff^{-1}(F)} \subset \overline{F} = F$.

$$\therefore \quad C_s[f^{-1}(F)] \subset f^{-1}(F).$$

Consequently, inverse of a closed set is semi-closed.

(g) $\longleftrightarrow$ (h): Suppose that (g) holds and let B be any subset of X^*. Replacing A by $f^{-1}(B)$ we get from (g)

$$f[C_s\{f^{-1}(B)\}] \subset \overline{ff^{-1}(B)} \subset \overline{B}$$

$$\text{i.e.} \quad C_s\{f^{-1}(B)\} \subset f^{-1}(\overline{B}).$$

If (h) holds, let $B = f(A)$ where A is a subset of X. Then

$$C_s(A) \subset C_s\{f^{-1}(B)\} \subset f^{-1}\{\overline{f(A)}\}.$$

$$\therefore \quad f[C_s(A)] \subset \overline{f(A)}.$$

This completes the proof of the theorem.

Remark. It may be seen from the following example that in Theorem 4, the property (a) will not imply any of the other properties (b)–(h) if open set, closed set, neighbourhood and closure are replaced respectively by semi-open set, semi-closed set, semi-neighbourhood and semi-closure.

Example. Let $f:(X, \Lambda) \to (X^*, \Lambda^*)$ be the identity mapping where

$$X = X^* = \{a, b, c, d\}$$
$$\Lambda = \Phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}; \{a, b, c\}$$
$$\Lambda^* = \Phi, X^*, \{a\}, \{a, c\}.$$

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[2] KELLEY J. L., *General Topology*., Van Nostrand, Princeton, N. J. 1955.

[3] LEVINE N., *Semi-open sets and semi-continuity in topological spaces*, « Amer. Math. Monthly », *70*, 36–41 (1963).