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On Fichera's transformation in the method of intermediate problems

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Analisi matematica. — *On Fichera's transformation in the method of intermediate problems.* Nota di WILLIAM STENGER (*), presentata (**) dal Socio Straniero A. WEINSTEIN.

RIASSUNTO. — Si stabilisce una relazione fra il determinante di Fichera e il determinante di Weinstein nella teoria dei problemi intermedi.

1. INTRODUCTION. – Among the many theoretical ramifications of the method of intermediate problems [1, 2, 3] appears the question of reducing a given problem of one type of intermediate problems to one of another type. Kuroda [4] and Fichera [2] have each given a transformation of the first to the second type for certain subclasses of problems. For more general problems, however, such reductions have not been achieved.

The purpose of the present note is to discuss Fichera's transformation and to derive a relationship between the corresponding pair of Weinstein determinants.

2. INTERMEDIATE PROBLEMS IN THE GENERAL CASE.—Let A be a self-adjoint operator defined on a dense subspace of a Hilbert space $\mathbf{H}$ having inner product (u, v). We denote by R_λ the resolvent of A, $R_\lambda = (A - \lambda I)^{-1}$. We assume that A is bounded below (or above) and that the lower (or upper) part of its spectrum consists of isolated eigenvalues each having finite multiplicity.

An *intermediate problem of the first type* is an eigenvalue problem having the form

$$Au - P_n Au = \lambda u , \tag{1}$$

$$P_n u = 0 , \tag{2}$$

where P_n is an orthogonal projection operator onto an n–dimensional subspace $\mathbf{P}_n$ of $\mathbf{H}$. The determination of the eigenvalues of (1, 2) is achieved via the *Weinstein determinant*

$$W_n(\lambda) = \det\{(R_\lambda p_i , p_k)\} \qquad (i , k = 1 , 2 , \cdots , n) \tag{3}$$

where $\{p_1 , p_2 , \cdots , p_n\}$ is a basis for $\mathbf{P}_n$.

An *intermediate problem of the second type* is a problem of having the form

$$Au + B_r u = \lambda u \tag{4}$$

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where B_r is a symmetric operator of rank $r < \infty$. Equation (4) can be written as

$$Au + \sum_{j=1}^{r} \alpha_j (u, q_j) q_j = \lambda u, \tag{5}$$

for a suitable choice of α_j's and q_j's, see [5, p. 161]. The spectrum of (5) is determined by the *modified Weinstein determinant* or *Weinstein–Aronszajn determinant*

$$V_r(\lambda) = \det \{ \delta_{ik} + \alpha_i (R_\lambda q_i, q_k) \} \qquad (i, k = 1, 2, \cdots, r). \tag{6}$$

While the matrix in (6) is non-symmetric, we can symmetrize it in an elementary way by multiplying the k^{th} column by α_k $(k = 1, 2, \cdots, r)$, see [6].

3. FICHERA'S TRANSFORMATION.— Following Fichera we now assume that A is a positive compact operator. We denote by $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and $u_1, u_2, \cdots$ the eigenvalues and corresponding eigenvectors of A. Using the square root, $A^{1/2}$, of A Fichera introduced a new problem

$$Av - A^{1/2} P_n A^{1/2} v = \mu v \tag{7}$$

and established the following connection between problems (1, 2) and (7), [2, p. 131].

THEOREM 1. *Every eigenvalue for problem* (1, 2) *is an eigenvalue for problem* (7). *Conversely, every non-zero eigenvalue for* (7) *is an eigenvalue for* (1, 2).

In view of the fact that $-A^{1/2} P_n A^{1/2}$ is a symmetric operator of finite rank, Theorem 1 provides a reduction of the first type of intermediate problems to the second in this case.

4. FUNCTIONAL EQUATION FOR DETERMINANTS.—The following relationship between two kinds of Weinstein determinants seems to be new.

THEOREM 2. *The determinants* (3) *and* (6) *corresponding to problems* (1, 2) *and* (7) *satisfy*

$$V_n(\lambda) = (-\lambda)^n W_n(\lambda) \tag{8}$$

for all λ.

Proof. Taking an orthonormal basis $\{p_1, p_2, \cdots, p_n\}$ for $\mathbf{P}_n$ we write (7) as

$$Av - \sum_{j=1}^{n} (A^{1/2} p_j, v) A^{1/2} p_j = \mu v.$$

Now determinant (6) is given by

$$V_n(\lambda) = \det \{\delta_{ik} - (R_\lambda A^{1/2} p_i, A^{1/2} p_k)\} \qquad (i, k = 1, 2, \cdots, n).$$

Note that here the matrix is symmetric since $\alpha_i = -1$ $(i = 1, 2, \cdots, n)$. Since A is a positive compact operator, for any real λ, $\lambda \neq \lambda_j$ $(j = 1, 2, \cdots)$ and for any $u \in \mathbf{H}$ we have

$$\begin{aligned} R_\lambda A^{1/2} v &= R_\lambda \sum_{j=1}^{\infty} \lambda_j^{1/2} (v, u_j) u_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{1/2} (v, u_j) R_\lambda u_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{1/2} (v, u_j) (\lambda_j - \lambda)^{-1} u_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{1/2} (v, [\lambda_j - \lambda]^{-1} u_j) u_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{1/2} (v, R_\lambda u_j) u_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{1/2} (R_\lambda v, u_j) u_j \\ &= A^{1/2} R_\lambda v . \end{aligned}$$

Therefore we can write

$$\begin{aligned} \delta_{ik} - (R_\lambda A^{1/2} p_i, A^{1/2} p_k) &= (p_i, p_k) - (R_\lambda A p_i, p_k) \\ &= (R_\lambda [A - \lambda I] p_i, p_k) - (R_\lambda A p_i, p_k) \\ &= -\lambda (R_\lambda p_i, p_k) \qquad (i, k = 1, 2, \cdots, n) \end{aligned}$$

so that we obtain

$$V_n(\lambda) = (-\lambda)^n W_n(\lambda) \tag{9}$$

for all λ not in the spectrum of A. However, since $V_n(\lambda)$ and $W_n(\lambda)$ are both meromorphic functions of λ, our equation (9) holds for all λ.

5. COMPARISON WITH KURODA'S TRANSFORMATION.—At first glance equation (8) looks strikingly similar to a result of Kuroda [4, p. 11]. However, these are two fundamentally different results as can be seen in the following.

Here it is no longer essential to take A to be a positive compact operator. Instead it is sufficient for A to be bounded. Kuroda considers the problem

$$(I - P_n) A (I - P_n) u = \lambda u \tag{10}$$

which is equivalent to (1, 2) on the subspace orthogonal to $\mathbf{P}_n$, see [2, p. 131]. Kuroda decomposes (10) into

$$Au - [(I - P_n) AP_n + P_n A] u = \lambda u . \tag{11}$$

We see that boundedness is necessary in (11). Otherwise AP_n would not be defined for an arbitrary choice of $\mathbf{P}_n$ and (11) would be meaningless. Let us also note that essentially the same decomposition of (10) was independently discovered by Fichera [2, p. 126]. Since the operator

$$(12) \qquad (I - P_n) AP_n + P_n A$$

is of finite rank, equation (11) is of the form (4). Surprisingly, however, the rank r of (12) is not in general n. Instead we have $n \leq r \leq 2n$. Therefore, the matrix leading to determinant (6) for (11) is $r \times r$ and *not* necessarily $n \times n$, see [7], while the matrices in (3) for (1, 2) and in (6) for (7) are *always* $n \times n$. In fact, putting $n = 1$ we have (3) given by a single element, namely

$$W(\lambda) = (R_\lambda p, p)$$

while (6) for (11) is given by the determinant of a 2×2 nonsymmetric matrix. In this case Kuroda's result [4, p. 11] has the form

$$(13) \quad W(\lambda) = (-\lambda)^{-1} \det \begin{Bmatrix} 1 - (R_\lambda Ap, p) + (Ap, p) W(\lambda) & -W(\lambda) \\ (R_\lambda Ap, Ap) - (Ap, p)(R_\lambda p, Ap) & -1 + (R_\lambda p, Ap) \end{Bmatrix}.$$

Note that $W(\lambda)$, which already provides all the information about the eigenvalues, suprisingly reappears in two elements of the non-symmetric matrix defining the determinant in the right-hand side of (13).

Summarizing, one could say that this attempt to get rid of $W(\lambda)$ by reducing (10) to (11) only results in the reappearance of two $W(\lambda)$'s accompanied by several other terms.

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