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On a class of integro-differential equations. Nota III

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Analisi matematica. — *On a class of integro-differential equations.* Nota III di Mehmet Namik Oğuztöreli (*) (**), presentata (***) dal Socio M. Picone.

Riassunto. — L'A. studia in questo lavoro la soluzione di un sistema di equazioni integro-differenziali a derivate parziali del secondo ordine dipendente da due parametri.

1. Introduction.

Consider the integro-differential system

$$(1.1)\quad \begin{cases} \lambda \dfrac{\partial^2 u(x,y)}{\partial x^2} + (\lambda - 1)\, u(x,y) = p_0(x,y) \\ \qquad + \mu \displaystyle\iint_R \{K_{11}(x,y;\xi,\eta)\, u(\xi,\eta) + K_{12}(x,y;\xi,\eta)\, v(\xi,\eta)\}\, d\xi\, d\eta \\ \lambda \dfrac{\partial^2 v(x,y)}{\partial y^2} + (\lambda - 1)\, v(x,y) = q_0(x,y) \\ \qquad + \mu \displaystyle\iint_R \{K_{21}(x,y;\xi,\eta)\, u(\xi,\eta) + K_{22}(x,y;\xi,\eta)\, v(\xi,\eta)\}\, d\xi\, d\eta \end{cases}$$

subject to the boundary conditions

$$(1.2)\quad \begin{cases} u(0,y) = u(1,y) = 0 & \text{for } 0 \le y \le 1, \\ v(x,0) = v(x,1) = 0 & \text{for } 0 \le x \le 1, \end{cases}$$

where $R = \{(x,y) \mid 0 \le x, y \le 1\}$, λ and μ are real parameters, $p_0(x,y)$, $q_0(x,y)$ and $K_{ij}(x,y;\xi,\eta)$, $i,j = 1,2$, are given real valued functions which are continuous on R, R and R×R respectively, and $u(x,y)$, $v(x,y)$ are the unknown functions. We assume that at least one of the functions $K_{12}(x,y;\xi,\eta)$ and $K_{21}(x,y;\xi,\eta)$ is not identically zero on R×R.

Boundary value problems of the form (1.1)–(1.2) occur in some min-max problems involving distributed parameter control systems (cfr. [1]). In this paper we investigate the continuous solutions of the boundary value

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problem (1.1)–(1.2) for $\lambda \neq \frac{1}{1-k^2\pi^2}$, $k = 1, 2, 3, \cdots$, and for sufficiently small μ. The results of the paper [2] will be extended.

The case $\lambda = \frac{1}{1-k^2\pi^2}$ will be discussed separately in a subsequent paper.

2. Solution for small μ.

In this section we establish a solution of the form

$$(2.1)\quad \begin{cases} u(x, y) = \sum_{n=0}^{\infty} \mu^n u_n(x, y) \quad , \quad u_n(0, y) = u_n(1, y) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, 2, \cdots \\ v(x, y) = \sum_{n=0}^{\infty} \mu^n v_n(x, y) \quad , \quad v_n(x, 0) = v_n(x, 1) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, 2, \cdots \end{cases}$$

for the boundary value problem (1.1)–(1.2). By substitution and comparison, we easily find the following recursive equations:

$$(2.2)\quad \begin{cases} \lambda \dfrac{\partial^2 u_n(x, y)}{\partial x^2} + (\lambda - 1)\, u_n(x, y) = p_n(x, y), \\ \lambda \dfrac{\partial^2 v_n(x, y)}{\partial y^2} + (\lambda - 1)\, v_n(x, y) = q_n(x, y), \end{cases} \qquad n = 0, 1, 2, \cdots$$

where

$$(2.3)\quad \begin{cases} p_n(x, y) = \displaystyle\iint_R \{ K_{11}(x, y; \xi, \eta)\, u_{n-1}(\xi, \eta) \\ \qquad\qquad + K_{12}(x, y; \xi, \eta)\, v_{n-1}(\xi, \eta)\}\, d\xi\, d\eta, \\ q_n(x, y) = \displaystyle\iint_R \{ K_{21}(x, y; \xi, \eta)\, u_{n-1}(\xi, \eta) \\ \qquad\qquad + K_{22}(x, y; \xi, \eta)\, v_{n-1}(\xi, \eta)\}\, d\xi\, d\eta, \end{cases}$$

for $n = 1, 2, 3, \cdots$. As in [2], we can easily verify that, for $\lambda \neq 0$ and $\lambda \neq \frac{1}{1-k^2\pi^2}$, $k = 1, 2, 3, \cdots$, we have

$$(2.4)\quad \begin{cases} u_0(x, y) = \dfrac{1}{\lambda} p_0(x, y) + \dfrac{\lambda - 1}{\lambda^2} \displaystyle\int_0^1 \Gamma\left(x, \sigma; \frac{\lambda - 1}{\lambda}\right) p_0(\sigma, y)\, d\sigma, \\ v_0(x, y) = \dfrac{1}{\lambda} q_0(x, y) + \dfrac{\lambda - 1}{\lambda^2} \displaystyle\int_0^1 \Gamma\left(y, \sigma; \frac{\lambda - 1}{\lambda}\right) q_0(x, \sigma)\, d\sigma \end{cases}$$

and the Fredholm integral equations with respect to $u_n(x, y)$ and $v_n(x, y)$:

$$(2.5)\quad \begin{cases} u_n(x, y) = -\frac{1}{\lambda}\iint\limits_R \{H_{11}(x, y; \xi, \eta)\, u_{n-1}(\xi, \eta) \\ \qquad + H_{12}(x, y; \xi, \eta)\, v_{n-1}(\xi, \eta)\}\, d\xi\, d\eta + \frac{\lambda-1}{\lambda}\int_0^1 G(x, \sigma)\, u_n(\sigma, y)\, d\sigma, \\ v_n(x, y) = -\frac{1}{\lambda}\iint\limits_R \{H_{21}(x, y; \xi, \eta)\, u_{n-1}(\xi, \eta) \\ \qquad + H_{22}(x, y; \xi, \eta)\, v_{n-1}(\xi, \eta)\}\, d\xi\, d\eta + \frac{\lambda-1}{\lambda}\int_0^1 G(y, \sigma)\, v_n(x, \sigma)\, d\sigma, \end{cases}$$

for $n = 1, 2, 3, \cdots$, where

$$(2.6)\quad G(t, \sigma) = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin k\pi t \sin k\pi\sigma}{k^2} = \begin{cases} t(1-\sigma) & \text{for } 0 \le t \le \sigma \le 1, \\ \sigma(1-t) & \text{for } 0 \le \sigma \le t \le 1, \end{cases}$$

$$(2.7)\quad \begin{cases} H_{1j}(x, y; \xi, \eta) = \int_0^1 G(x, \sigma)\, K_{1j}(\sigma, y; \xi, \eta)\, d\sigma \\ H_{2j}(x, y; \xi, \eta) = \int_0^1 G(y, \sigma)\, K_{2j}(x, \sigma; \xi, \eta)\, d\sigma, \end{cases} \qquad (j = 1, 2)$$

and $\Gamma(t, \sigma; \nu)$ is the resolvent of the kernel $G(t, \sigma)$:

$$(2.8)\quad \Gamma(t, \sigma; \nu) = 2\sum_{k=1}^{\infty}\frac{\sin k\pi t \sin k\pi\sigma}{k^2\pi^2 - \nu}\,.$$

Thus

$$(2.9)\quad \begin{cases} u_n(x, y) = -\frac{1}{\lambda}\iint\limits_R \Big\{\Omega_{11}\Big(x, y; \xi, \eta; \frac{\lambda-1}{\lambda}\Big)\, u_{n-1}(\xi, \eta) \\ \qquad + \Omega_{12}\Big(x, y; \xi, \eta; \frac{\lambda-1}{\lambda}\Big)\, v_{n-1}(\xi, \eta)\Big\}\, d\xi\, d\eta \\ v_n(x, y) = -\frac{1}{\lambda}\iint\limits_R \Big\{\Omega_{21}\Big(x, y; \xi, \eta; \frac{\lambda-1}{\lambda}\Big)\, u_{n-1}(\xi, \eta) \\ \qquad + \Omega_{22}\Big(x, y; \xi, \eta; \frac{\lambda-1}{\lambda}\Big)\, v_{n-1}(\xi, \eta)\Big\}\, d\xi\, d\eta \end{cases}$$

for $n = 1, 2, 3, \cdots$, where

$$(2.10)\quad \begin{cases} \Omega_{1j}(x, y; \xi, \eta; \nu) = H_{1j}(x, y; \xi, \eta) + \nu \int_0^1 \Gamma(x, \sigma; \nu) H_{1j}(\sigma, y; \xi, \eta) \, d\sigma, \\ \Omega_{2j}(x, y; \xi, \eta; \nu) = H_{2j}(x, y; \xi, \eta) + \nu \int_0^1 \Gamma(y, \sigma; \nu) H_{2j}(x, \sigma; \xi, \eta) \, d\sigma, \end{cases} \qquad j = 1, 2.$$

Clearly, each $u_n(x, y)$ and $v_n(x, y)$ is uniquely determined by Eqs. (2.4) and (2.9).

We now establish the convergence of the two series in (2.1). To this end, we first introduce the notation

$$(2.11)\qquad \|w\| = \max_R |w(x, y)|$$

for any $w = w(x, y)$ defined on R, and

$$(2.12)\qquad \begin{aligned} &\gamma = \frac{1}{8} \max \{\|p_0\|, \|q_0\|\} \quad , \quad \nu = \frac{\lambda - 1}{\lambda} \\ &H = \sum_{i,j=1}^{2} \left\| \iint_R |H_{ij}(x, y; \xi, \eta)| \, d\xi \, d\eta \right\|, \\ &K = \sum_{i,j=1}^{2} \left\| \iint_R |K_{ij}(x, y; \xi, \eta)| \, d\xi \, d\eta \right\|, \\ &\Gamma_\nu = \left\| \int_0^1 |\Gamma(x, \sigma; \nu)| \, d\sigma \right\|, \\ &\Omega_\nu = \sum_{i,j=1}^{2} \left\| \iint_R |\Omega_{ij}(x, y; \xi, \eta; \nu)| \, d\xi \, d\eta \right\|. \end{aligned}$$

We can easily verify that

$$(2.13)\qquad \begin{cases} \|u_n\| \leq \left(\frac{\Omega_\nu}{|\lambda|}\right)^n \frac{1 + |\nu| \Gamma_\nu}{|\lambda|} \gamma, \\ \|v_n\| \leq \left(\frac{\Omega_\nu}{|\lambda|}\right)^n \frac{1 + |\nu| \Gamma_\nu}{|\lambda|} \gamma, \end{cases} \qquad n = 1, 0, 2, \cdots$$

$$(2.14)\qquad \begin{cases} \left\|\frac{\partial^2 u_0}{\partial x^2}\right\| \leq \frac{8 + |\nu| + \nu^2 \Gamma_\nu}{|\lambda|} \gamma, \\ \left\|\frac{\partial^2 v_0}{\partial y^2}\right\| < \frac{8 + |\nu| + \nu^2 \Gamma_\nu}{|\lambda|} \gamma, \end{cases}$$

and

$$(2.15)\quad \begin{cases} \left\| \dfrac{\partial^2 u_n}{\partial x^2} \right\| \leq \dfrac{(1+|\nu|\Gamma_\nu)(K+|\nu|\Omega_\nu)}{\lambda^2} \left(\dfrac{\Omega_\nu}{|\lambda|}\right)^{n-1} \gamma , \\ \left\| \dfrac{\partial^2 v_n}{\partial y^2} \right\| < \dfrac{(1+|\nu|\Gamma_\nu)(K+|\nu|\Omega_\nu)}{\lambda^2} \left(\dfrac{\Omega_\nu}{|\lambda|}\right)^{n-1} \gamma . \end{cases} \qquad n = 1, 2, 3, \cdots .$$

The above inequalities assure the absolute and uniform convergence of the four series. Similar results can be easily established for the first derivatives series.

$$\sum_{n=0}^{\infty} \mu^n u_n(x, y) \quad , \quad \sum_{n=0}^{\infty} \mu^n v_n(x, y) \quad , \quad \sum_{n=0}^{\infty} \mu^n \frac{\partial^2 u_n(x, y)}{\partial x^2} \quad , \quad \sum_{n=0}^{\infty} \mu^n \frac{\partial^2 v_n(x, y)}{\partial y^2}$$

on the square R for

$$(2.16)\qquad |\mu| < \frac{|\lambda|}{\Omega_\nu} .$$

Thus, we may state the following:

THEOREM. *For* $\lambda \neq 0$, $\lambda \neq \lambda_k$ *and for* μ *satisfying the inequality* (2.16), *there exists a unique solution* $\{u(x, y), v(x, y)\}$, *given by* (2.1), (2.4) *and* (2.9), *of the boundary value problem* (1.1)-(1.2).

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