Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giovanni Prouse

On a non—linear mixed problem for the Navier-Stokes equations. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **48** (1970), n.3, p. 293–296.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1970_8_48_3_293_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1970.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 marzo 1970

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On a non-linear mixed problem for the Navier-Stokes equations.* Nota III di GIOVANNI PROUSE [(*)], presentata [(**)] dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dimostra un teorema di unicità della soluzione del problema posto nella Nota I.

4. Let us prove the following uniqueness theorem.

Let $\vec{u}(t), \vec{v}(t)$ *be two solutions on* $0 \vdash\!\dashv T$ *of the Navier-Stokes system* (in the sense indicated at § 3), *corresponding to the same initial and boundary conditions. Then* $\vec{u}(t) \equiv \vec{v}(t)$.

We observe, first of all, that, setting $\vec{w}(t) = \vec{u}(t) - \vec{v}(t)$, the function $\vec{w}(t)$ satisfies, by (3.15), the equation

$$\int_0^T \{-(\vec{w}(t), \vec{h}'(t))_{V_0} + \mu(\vec{w}(t), \vec{h}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{h}(t)) - \tag{4.1}$$

$$- \vec{b}(v(t), \vec{v}(t), \vec{h}(t))\}\, dt =$$

$$= \int_0^T \int_0^k \left\{ \left(-\frac{1}{2} u_1^2(0, x_2, t) + \frac{1}{2} v_1^2(0, x_2, t)\right) h_1(0, x_2, t) + \right.$$

$$\left. + \left(\frac{1}{2} u_1^2(l, x_2, t) - \frac{1}{2} v_1^2(l, x_2, t)\right) h_1(l, x_2, t) \right\} dx_2\, dt -$$

(*) Istituto Matematico del Politecnico di Milano.

Lavoro eseguito nell'ambito dell'attività del Contratto di Ricerca « Equazioni Funzionali » del Comitato per la Matematica del C.N.R.

(**) Nella seduta del 13 dicembre 1969.

$$-\int_0^T\int_0^l \{\beta_1(x_1,t)(|u_2(x_1,0,t)|\,u_2(x_1,0,t)-|v_2(x_1,0,t)|\,v_2(x_1,0,t))\,h_2(x_1,0,t)+$$

$$+\beta_2(x_1,t)(|u_2(x_1,k,t)|\,u_2(x_1,k,t)-|v_2(x_1,k,t)|\,v_2(x_1,k,t))\,h_2(x_1,k,t)\}\,dx_1\,dt.$$

Repeating the procedure followed to prove (3.37), it can be shown that, if $\vec{u}(t)$ is a solution on $0 \vdash\!\dashv T$, then $\vec{u}'(t) \in L^{4/3}(0 \vdash\!\dashv T\,;V_{\sigma-1})$ (see also Baiocchi [8], Lions [11])(1). By a theorem of Strauss [12] we obtain therefore, setting in (4.1) $\vec{h}(t) = \vec{w}(t)$ and bearing in mind that $\vec{w}(0) = 0$,

$$(4.2)\qquad \frac{1}{2}\|\vec{w}(t)\|_{V_0}^2 + \mu\int_0^t \|\vec{w}(\eta)\|_{V_1}^2\,d\eta + \int_0^t (b(\vec{u}(\eta),\vec{u}(\eta),\vec{w}(\eta)) -$$

$$- b(\vec{v}(\eta),\vec{v}(\eta),\vec{w}(\eta)))\,d\eta =$$

$$=\int_0^t\int_0^k \frac{1}{2}\{(-u_1^2(0,x_2,\eta)+v_1^2(0,x_2,\eta))\,w_1(0,x_2,\eta)+$$

$$+(u_1^2(l,x_2,\eta)-v_1^2(l,x_2,\eta))\,w_1(l,x_2,\eta)\}\,dx_2\,d\eta -$$

$$-\int_0^t\int_0^l\{\beta_1(x_1,\eta)(|u_2(x_1,0,\eta)|\,u_2(x_1,0,\eta)-|v_2(x_1,0,\eta)|\,v_2(x_1,0,\eta))\,w_2(x_1,0,\eta)+$$

$$+\beta_2(x_1,\eta)(|u_2(x_1,k,\eta)|\,u_2(x_1,k,\eta)-|v_2(x_1,k,\eta)|\,v_2(x_1,k,\eta))\,w_2(x_1,k,\eta)\}\,dx_1\,d\eta.$$

Observe that the last integral on the right hand side of (4.2) is, by the assumptions made, ≥ 0. Hence

$$(4.3)\qquad \|\vec{w}(t)\|_{V_0}^2 + 2\mu\int_0^t\|\vec{w}(\eta)\|_{V_1}^2\,d\eta \leq 2\int_0^t |\,b(\vec{u}(\eta),\vec{u}(\eta),\vec{w}(\eta)) -$$

$$- b(\vec{v}(\eta),\vec{v}(\eta),\vec{w}(\eta))\,|\,d\eta +$$

$$+\int_0^t\int_0^k\{|\,u_1(0,x_2,\eta)+v_1(0,x_2,\eta)\,|\,w_1^2(0,x_2,\eta)+$$

$$+|\,u_1(l,x_2,\eta)+v_1(l,x_2,\eta)\,|\,w_1^2(l,x_2,\eta)\}\,dx_2\,d\eta\,.$$

On the other hand,

$$(4.4)\qquad |\,b(\vec{u},\vec{u},\vec{w}) - b(\vec{v},\vec{v},\vec{w})\,| \leq |\,b(\vec{w},\vec{v},\vec{w})\,| + |\,b(\vec{u},\vec{w},\vec{w})\,|$$

(1) Per tutte le citazioni bibliografiche vedi Nota I, questi « Rendiconti », vol. XLVIII, fasc. 1, p. 26 (1970).

and, by an inequality of Ladyzenskaja,

$$(4.5)\qquad |b(\vec{w},\vec{v},\vec{w})| \le \|\vec{w}\|_{L^4(\Omega)} \|\vec{v}\|_{H^1(\Omega)} \|\vec{w}\|_{L^4(\Omega)} \le \sqrt{2}\, \|\vec{v}\|_{H^1(\Omega)} \|\vec{w}\|_{L^2(\Omega)} \|\vec{w}\|_{H^1(\Omega)} \le$$
$$\le \frac{\mu}{4} \|\vec{w}\|^2_{V_1} + c_\mu \|\vec{v}\|^2_{V_1} \|\vec{w}\|^2_{V_0}.$$

Analogously, we have

$$(4.6)\qquad |b(\vec{u},\vec{w},\vec{w})| \le \|\vec{u}\|_{L^4(\Omega)} \|\vec{w}\|_{H^1(\Omega)} \|\vec{w}\|_{L^4(\Omega)} \le \sqrt{2}\, \|\vec{u}\|_{L^4(\Omega)} \|\vec{w}\|^{3/2}_{H^1(\Omega)} \|\vec{w}\|^{1/2}_{L^2(\Omega)} \le$$
$$\le \frac{\mu}{4} \|\vec{w}\|^2_{V_1} + c'_\mu \|\vec{u}\|^4_{L^4(\Omega)} \|\vec{w}\|^2_{V_0} \le \frac{\mu}{4} \|\vec{w}\|^2_{V_1} + c''_\mu \|\vec{u}\|^2_{V_0} \|\vec{u}\|^2_{V_1} \|\vec{w}\|^2_{V_0}.$$

Moreover, by the usual embedding theorems, bearing in mind that $\vec{u}(t), \vec{v}(t)$ are V_σ–bounded, we obtain, since $\sigma > 0$,

$$(4.7)\qquad \int_0^k \{ |u_1(0,x_2,t) + v_1(0,x_2,t)|\, w_1^2(0,x_2,t) +$$
$$+ |u_1(l,x_2,t) + v_1(l,x_2,t)|\, w_1^2(l,x_2,t) \}\, dx_2 \le$$
$$\le (\|\gamma_0 \vec{u}(t)\|_{L^2(\Gamma)} + \|\gamma_0 \vec{v}(t)\|_{L^2(\Gamma)}) \|\gamma_0 \vec{w}(t)\|^2_{L^4(\Gamma)} \le$$
$$\le c_1 (\|\vec{u}(t)\|_{H^{(1+\sigma)/2}(\Omega)} + \|\vec{v}(t)\|_{H^{(1+\sigma)/2}(\Omega)}) \|\vec{w}(t)\|_{H^{3/4}(\Omega)} \le$$
$$\le c_2 (\|\vec{u}(t)\|^{1/2}_{V_\sigma} \|\vec{u}(t)\|^{1/2}_{V_1} + \|\vec{v}(t)\|^{1/2}_{V_\sigma} \|\vec{v}(t)\|^{1/2}_{V_1}) \|\vec{w}(t)\|^{1/2}_{V_0} \|\vec{w}(t)\|^{3/2}_{V_1} \le$$
$$\le c_3 (\|\vec{u}(t)\|^{1/2}_{V_1} + \|\vec{v}(t)\|^{1/2}_{V_1}) \|\vec{w}(t)\|^{1/2}_{V_0} \|\vec{w}(t)\|^{3/2}_{V_1} \le$$
$$\le c_3 (\|\vec{u}(t)\|^{1/2}_{V_1} + \|\vec{v}(t)\|^{1/2}_{V_1}) \Big[\frac{\mu}{c_3} (\|\vec{u}(t)\|^{1/2}_{V_1} + \|\vec{v}(t)\|^{1/2}_{V_1})^{-1} \|\vec{w}(t)\|^2_{V_1} +$$
$$+ \frac{c_3^3}{\mu^3} (\|\vec{u}(t)\|^{1/2}_{V_1} + \|\vec{v}(t)\|^{1/2}_{V_1})^3 \|\vec{w}(t)\|^2_{V_0} \Big] \le$$
$$\le \mu \|\vec{w}(t)\|^2_{V_1} + c_4 (\|\vec{u}(t)\|^2_{V_1} + \|\vec{v}(t)\|^2_{V_1}) \|\vec{w}(t)\|^2_{V_0}.$$

Hence, by (4.4), (4.5), (4.6), (4.7),

$$(4.8)\qquad 2\, |b(\vec{u}(t),\vec{u}(t),\vec{w}(t)) - b(\vec{v}(t),\vec{v}(t),\vec{w}(t))| +$$
$$+ \int_0^k \{ |u_1(0,x_2,t) + v_1(0,x_2,t)|\, w_1^2(0,x_2,t) +$$
$$+ |u_1(l,x_2,t) + v_1(l,x_2,t)|\, w_1^2(l,x_2,t) \}\, dx_2 \le$$
$$\le 2\mu \|\vec{w}(t)\|^2_{V_1} + c_5 (\|\vec{u}(t)\|^2_{V_1} + \|\vec{v}(t)\|^2_{V_1}) \|\vec{w}(t)\|^2_{V_0}.$$

Since $\vec{u}(t), \vec{v}(t) \in L^2(0 \vdash\dashv T; V_1)$, we have

$$\|\vec{u}(t)\|_{V_1}^2 + \|\vec{v}(t)\|_{V_1}^2 = \chi(t) \in L^1(0 \vdash\dashv T). \tag{4.9}$$

Introducing (4.8), (4.9) into (4.3) we obtain

$$\|\vec{w}(t)\|_{V_0}^2 \leq c_5 \int_0^t \chi(\eta) \|\vec{w}(\eta)\|_{V_0}^2 d\eta,$$

from which follows that $\|\vec{w}(t)\|_{V_0} = 0$.

The theorem is therefore proved.

5. By a similar procedure to that followed in the preceding paragraphs it is possible to give existence and uniqueness theorems for the solutions of problems which are slightly different from the one so far considered.

We can in this way prove a uniqueness and existence (in the small) theorem of the solution satisfying the initial condition (3.1) and the boundary conditions (3.4), (3.5), (3.6) and

$$\begin{aligned} p(0, x_2, t) &= \alpha_1(x_2, t) \\ p(l, x_2, t) &= \alpha_2(x_2, t) \end{aligned} \tag{5.1}$$

or

$$\begin{aligned} u_1(0, x_2, t) &= \alpha_1(x_2, t) \\ u_1(l, x_2, t) &= \alpha_2(x_2, t). \end{aligned} \tag{5.2}$$

It is, in other words, possible to assign on the initial and final sections the values of the pressure or of the velocity.

If, instead of system (1.1), we consider the linearized Navier-Stokes system

$$\begin{cases} \dfrac{\partial u_j}{\partial t} - \mu \Delta u_j + \dfrac{\partial p}{\partial x_j} = f_j & (j = 1, 2) \\ \displaystyle\sum_{k=1}^{2} \frac{\partial u_k}{\partial x_k} = 0, \end{cases} \tag{5.3}$$

then the existence and uniqueness of a solution satisfying the initial condition (3.1) and the boundary conditions (3.4), (3.5), (3.6), (5.1) or (3.4), (3.5), (3.6), (5.2) can be proved in the large, that is on the whole interval $0 \vdash\dashv T$.

Observe, finally, that, if we consider the linearized system, conditions (3.4), (3.5) may be substituted by the following, more general, ones

$$\begin{aligned} p(x_1, 0, t) &= -\beta_1(x_1, t)\, \rho_1(u_2(x_1, 0, t)), \\ p(x_1, k, t) &= \beta_2(x_1, t)\, \rho_2(u_2(x_1, k, t)), \end{aligned} \tag{5.4}$$

where $\rho_i(\xi)$ $(i = 1, 2)$ are continuous functions of $\xi \in \overline{-\infty \;\; \infty}$, non-decreasing, with $\rho_i(0) = 0$ and such that

$$\nu_1 |\xi|^\sigma \leq \xi \rho_i(\xi) \leq \nu_2 |\xi|^\sigma,$$

ν_1 and ν_2 being positive constants and $\sigma \geq 1$.