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On Misra's covariant differentiation in a Fins1er Space

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Geometria differenziale. — *On Misra's covariant differentiation in a Finsler Space.* Nota di R. B. MISRA e K. S. PANDE, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — In relazione a cambiamenti proiettivi (che involgono una funzione di punto e direzione) in uno spazio di Sinsler il Misra aveva introdotto una connessione (indicata qui col suo nome) diversa da quelle di Berwald e Cartan. In questa Nota vengono date varie formule di commutazione fra la derivata covariante fatta con la connessione di Misra ed altre.

A projective change preserving the invariance of Berwald's curvature tensor has been considered by one of the present authors [3]. It has been proved therein the vanishing of the covariant derivative of the vector $\dot{\partial}_h P$ for the special connection parameters G^i_{jk} (1) is the necessary and sufficient condition to have the said invariance. In the present paper we define the covariant derivative of any vector-field for these connection parameters and study the further aspects of this differentiation. The commutation formulae involving this covariant differentiation with various processes of differentiation such as (i) the partial differentiation with respect to (w.r.t.) $\dot{x}^h$'s, (ii) the Cartan covariant differentiation w.r.t. $\dot{x}^h$'s, and (iii) the Lie differentiation have been derived. It is worth noting that these results are similar to those obtained by associating the Berwald covariant differentiation with the said processes of differentiation. It will be noted that this covariant differentiation does not possess all the characteristics which the Berwald covariant differentiation does.

1. – PRELIMINARIES.

Let F_n be an n-dimensional Finsler space in which the metric function $F(x^i, \dot{x}^i)$ (2) satisfies the requisite conditions. The entities given by

$$(1.1) \qquad g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2 \quad , \quad g^{ij}(x, \dot{x})\, g_{jk}(x, \dot{x}) = \delta^i_k \, , \qquad (\dot{\partial}_i = \partial/\partial\dot{x}^i),$$

constitute the metric tensor of the space which is symmetric and positively homogeneous of degree zero in the $\dot{x}^i$'s. Defining the functions

$$(1.2) \qquad G^i(x, \dot{x}) = \frac{1}{4} g^{ih}\{\partial_j g_{kh} + \partial_k g_{hj} - \partial_h g_{jk}\}\, \dot{x}^j \dot{x}^k , \qquad (\partial_j \equiv \partial/\partial x^j),$$

(*) Nella seduta del 14 febbraio 1970.

(1) Henceforth we shall call them as Misra's connection parameters and the corresponding covariant derivative as Misra's covariant derivative.

(2) The indices $i, j, k, \ldots$ run from 1 to n. The line-elements $(x^i, \dot{x}^i)$ will be briefly denoted by $(x, \dot{x})$.

Berwald introduced the connection parameters G^i_{jk} and the covariant derivative $\mathfrak{B}_k X^i$ of a vector-field $X^i(x, \dot{x})$:

$$G^i_{jk}(x, \dot{x}) = \dot{\partial}_j G^j_k = \dot{\partial}_j \dot{\partial}_k G^i, \tag{1.3}$$

$$\mathfrak{B}_k X^i = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}. \tag{1.4}$$

Clearly the functions G^i, G^i_j and G^i_{jk} are positively homogeneous of degree 2, 1, 0 respectively in the $\dot{x}^i$'s. Consequently they satisfy

$$G^i_{jk} \dot{x}^j = G^i_k \quad , \quad G^i_k \dot{x}^k = 2G^i. \tag{1.5}$$

Introducing the tensors

$$2C_{ijk}(x, \dot{x}) \stackrel{\text{def}}{=} \dot{\partial}_i g_{jk} \quad , \quad C^h_{ik}(x, \dot{x}) \stackrel{\text{def}}{=} g^{hj} C_{ijk}, \tag{1.6}$$

which are symmetric in their lower indices we have

$$C_{ijk} \dot{x}^k = 0 = C^h_{ik} \dot{x}^k. \tag{1.7}$$

For these entities the covariant derivative of a vector $X^i(x, \dot{x})$ w.r.t. $\dot{x}^h$ has been defined by [[2], p. 187]

$$\dot{\nabla}_h X^i = \dot{\partial}_h X^i + C^i_{hr} X^r. \tag{1.8}$$

2. – Misra's covariant differentiation.

Misra [[3], eqn. (4.5)] defined the connection parameters

$$\hat{G}^i_{jk}(x, \dot{x}) = G^i_{jk} - \delta^i_{(j} \dot{\partial}_{k)} P - \frac{1}{2} \dot{x}^i \dot{\partial}_j \dot{\partial}_k P, \tag{2.1}$$

where $P(x, \dot{x})$ is an arbitrary scalar function being positively homogeneous of degree one in $\dot{x}^i$'s. Thus, $\hat{G}^i_{jk}$ are also positively homogeneous functions of degree zero in $\dot{x}^i$'s and possess the same transformation law as G^i_{jk}. Analogous to (1.3) if, however, we put

$$\hat{G}^i_{jk} = \dot{\partial}_j \hat{G}^i_k \quad , \quad \hat{G}^i_k = \dot{\partial}_k \hat{G}^i, \tag{2.2}$$

whence the functions $\hat{G}^i(x, \dot{x})$ and $\hat{G}^i_k(x, \dot{x})$ are positively homogeneous of degree two and one respectively, we would have

$$\hat{G}^i_{jk} \dot{x}^j = \hat{G}^i_k \quad , \quad \hat{G}^i_k \dot{x}^k = 2\hat{G}^i \tag{2.3}$$

in accordance with (1.5). Consequently the explicit expressions for the functions $\hat{G}^i_k$ and $\hat{G}^i$ are given by

$$\hat{G}^i_k = G^i_k - \frac{1}{2}(\dot{x}^i \dot{\partial}_k P + P\delta^i_k) = G^i_k - \frac{1}{2} \dot{\partial}_k (P\dot{x}^i), \tag{2.4}$$

and

$$\hat{G}^i = G^i - \frac{1}{2} P\dot{x}^i. \tag{2.5}$$

Now we define the covariant derivative for these connection parameters in the manner analogous to (1.4). Denoting by $\mathfrak{M}_k X^i$ the corresponding covariant derivative of the vector-field $X^i(x, \dot{x})$ we thus have

$$\mathfrak{M}_k X^i = \partial_k X^i - (\dot{\partial}_j X^i)\hat{G}^j_k + X^j \hat{G}^i_{jk}. \tag{2.6}$$

The following theorem establishes the relationship of this covariant derivative with that of Berwald:

THEOREM 2.1. – *The covariant differential operators* $\mathfrak{M}_k$ *and* $\mathfrak{B}_k$ *are connected by*

$$2(\mathfrak{M}_k - \mathfrak{B}_k) X^i = (\dot{\partial}_j X^i)\,\dot{\partial}_k (P\dot{x}^j) - X^j \dot{\partial}_j \dot{\partial}_k (P\dot{x}^i). \tag{2.7}$$

Proof. Putting for G^i_{jk} and $\hat{G}^j_k$ from (2.1) and (2.4), and using (1.4) the equation (2.6) immediately reduces to (2.7).

Noting the homogeneity properties of the functions P and F we may derive, from the above theorem, the covariant derivatives of $\dot{x}^i$, $F(x, \dot{x})$ and the unit vector-field $l^i(x, \dot{x}) \equiv \dot{x}^i/F$. Thus we would have

COROLLARY 2.1. – *Like Berwald's covariant derivative Misra's covariant derivative of* $\dot{x}^i$ *also vanishes.*

COROLLARY 2.2. – *Misra's covariant derivative of the metric function is given by*

$$\mathfrak{M}_k F = \dot{\partial}_k (PF). \tag{2.8}$$

COROLLARY 2.3. – *Misra's covariant derivative of the unit vector* l^i *is given by*

$$\mathfrak{M}_k l^i = -\frac{1}{F} l^i \dot{\partial}_k (PF). \tag{2.9}$$

NOTE 2.1. – *In contrast with Berwald's covariant derivative Misra's covariant derivative of* F *and* l^i *does not vanish.*

The derivation of (2.7) may be generalised to any arbitrary tensor. Let $T^{\cdots}_{\cdots}(x, \dot{x})$ be a tensor of rank (p, q). Its covariant derivative is given by

$$2(\mathfrak{M}_k - \mathfrak{B}_k) T^{\cdots}_{\cdots} = (\dot{\partial}_r T^{\cdots}_{\cdots})\dot{\partial}_k (P\dot{x}^r) - \tag{2.10}$$

$$- \sum_{\alpha} T^{\cdots r \cdots}_{\cdots\cdots}\dot{\partial}_r \dot{\partial}_k (P\dot{x}^{i_\alpha}) + \sum_{\beta} T^{\cdots\cdots}_{\cdots r \cdots}\dot{\partial}_{j_\beta}\dot{\partial}_k (P\dot{x}^r).$$

Also, for the metric tensor g_{jk} we have, in view of (1.7), that

$$\mathfrak{M}_i g_{jk} = \mathfrak{B}_i g_{jk} + PC_{ijk} + g_{h(j}\dot{\partial}_{k)}\dot{\partial}_i (P\dot{x}^h). \tag{2.11}$$

Thus, it follows from (2.11) that $\mathfrak{M}_i g_{jk}$ does not, in general, vanish. Therefore we have

THEOREM 2.2. – *Like Berwald's connection Misra's connection is also not metric.*

3. – The commutation formulae.

It is well known that the Berwald's operator $\mathcal{B}_k$ commutes with the partial differential operator $\dot{\partial}_h$ according to [[1], eqn. (4.6.11a)].

$$(\mathcal{B}_k \dot{\partial}_h - \dot{\partial}_h \mathcal{B}_k)\, X^i = - G^i_{khr}\, X^r, \tag{3.1}$$

where $G^i_{hkr}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}\, G^i_{kh}$ is a tensor-field symmetric in all the lower indices. Also it may be seen by the ways analogous to those leading to the equations (2.1 a) of [4] and (4.7) of [5] that the Berwald operator $\mathcal{B}_k$ commutes with $\dot{\nabla}_h$ and the Lie operator $\mathfrak{L}$ according to

$$(\mathcal{B}_k \dot{\nabla}_h - \dot{\nabla}_h \mathcal{B}_k)\, X^i = C^r_{kh}\, \mathcal{B}_r\, X^i + (\mathcal{B}_k\, C^i_{hr} - G^i_{khr})\, X^r, \tag{3.2}$$

and

$$(\mathfrak{L} \mathcal{B}_k - \mathcal{B}_k \mathfrak{L})\, X^i = X^j\, \mathfrak{L} G^i_{jk} - (\dot{\partial}_j\, X^i)\, \mathfrak{L} G^j_k . \tag{3.3}$$

In the same way it may be desirable to have the corresponding commutation formulae when the operator $\mathcal{B}_k$ is replaced by $\mathfrak{M}_k$. Therefore, in the following theorems we derive the commutation rules of the operator $\mathfrak{M}_k$ with the operators $\dot{\partial}_h$, $\dot{\nabla}_h$ and $\mathfrak{L}$.

THEOREM 3.1. – *For a vector-field* $X^i(x, \dot{x})$ *the operators* $\mathfrak{M}_k$ *and* $\dot{\partial}_h$ *commute according to*

$$(\mathfrak{M}_k \dot{\partial}_h - \dot{\partial}_h \mathfrak{M}_k)\, X^i = - \hat{G}^i_{khr}\, X^r, \tag{3.4}$$

where

$$\hat{G}^i_{khr}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_r\, \hat{G}^i_{kh} . \tag{3.5}$$

Proof. – Applying (2.10) for the tensor $\dot{\partial}_h X^i$ we may obtain

$$\begin{aligned} 2\, (\mathfrak{M}_k \dot{\partial}_h - \mathcal{B}_k \dot{\partial}_h)\, X^i &= (\dot{\partial}_r\, \dot{\partial}_h\, X^i)\, \dot{\partial}_k\, (P\dot{x}^r) - \\ &- (\dot{\partial}_h\, X^r)\, \dot{\partial}_r \dot{\partial}_k\, (P\dot{x}^i) + (\dot{\partial}_r\, X^i)\, \dot{\partial}_h\, \dot{\partial}_k\, (P\dot{x}^r) . \end{aligned} \tag{3.6}$$

Also, differentiating (2.7) w.r.t. $\dot{x}^h$ we get

$$\begin{aligned} 2\, (\dot{\partial}_h\, \mathfrak{M}_k - \dot{\partial}_h \mathcal{B}_k)\, X^i &= (\dot{\partial}_h\, \dot{\partial}_r\, X^i)\, \dot{\partial}_k\, (P\dot{x}^r) + (\dot{\partial}_r\, X^i)\, \dot{\partial}_h\, \dot{\partial}_k\, (P\dot{x}^r) - \\ &- (\dot{\partial}_h\, X^r)\, \dot{\partial}_r \dot{\partial}_k\, (P\dot{x}^i) - X^r\, \dot{\partial}_h\, \dot{\partial}_k\, \dot{\partial}_r\, (P\dot{x}^i) . \end{aligned} \tag{3.7}$$

Subtracting (3.7) from (3.6), and using (3.1) it follows that

$$(\mathfrak{M}_k \dot{\partial}_h - \dot{\partial}_h \mathfrak{M}_k)\, X^i = - \left\{ G^i_{khr} - \frac{1}{2}\, \dot{\partial}_k \dot{\partial}_h \dot{\partial}_r\, (P\dot{x}^i) \right\} X^r .$$

For (2.2), (2.4) and (3.5) this formula simplifies to (3.4).

The derivation of (3.4) may be generalised for any arbitrary tensor $T^{\cdots}_{\cdots}(x, \dot{x})$ in the form:

$$(3.8) \qquad (\mathfrak{M}_k \dot{\partial}_h - \dot{\partial}_h \mathfrak{M}_k)\, T^{\cdots}_{\cdots} = -\sum_{\alpha} \hat{G}^{i_\alpha}_{khr}\, T^{\cdots r \cdots}_{\cdots\cdots} + \sum_{\beta} \hat{G}^{r}_{khj_\beta}\, T^{\cdots\cdots}_{\cdots r \cdots}\,.$$

Further, as a particular case, we may have the following deduction of the above theorem.

COROLLARY 3.1. – *For the vector-fields $\dot{x}^i$ and l^i the operators $\mathfrak{M}_k$ and $\dot{\partial}_h$ are commutative.*

Proof. – Since the functions $\hat{G}^i_{kh}$ are homogeneous of degree zero in $\dot{x}^i$'s we have from (3.5)

$$(3.9) \qquad \hat{G}^i_{khr}\, \dot{x}^r = 0 = \hat{G}^i_{khr}\, l^r.$$

Hence the corollary follows immediately from (3.4).

THEOREM 3.2. – *For a vector-field $X^i(x, \dot{x})$ the operators $\mathfrak{M}_k$ and $\dot{\nabla}_h$ commute according to*

$$(3.10) \qquad (\mathfrak{M}_k \dot{\nabla}_h - \dot{\nabla}_h \mathfrak{M}_k)\, X^i = C^r_{kh}\, \mathfrak{M}_r\, X^i + (\mathfrak{M}_k C^i_{hr} - \hat{G}^i_{khr})\, X^r.$$

Proof. – Operating (1.8) by $\mathfrak{M}_k$ we get

$$(3.11) \qquad \mathfrak{M}_k \dot{\nabla}_h X^i = \mathfrak{M}_k \dot{\partial}_h X^i + (\mathfrak{M}_k C^i_{hr})\, X^r + C^i_{hr}\, \mathfrak{M}_k\, X^r.$$

Also, an application of the formula (1.8) for the tensor-field $\mathfrak{M}_k X^i$ yields

$$(3.2) \qquad \dot{\nabla}_h \mathfrak{M}_k X^i = \dot{\partial}_h \mathfrak{M}_k X^i - (\mathfrak{M}_r X^i)\, C^r_{kh} + C^i_{hr}\, \mathfrak{M}_k X^r\,.$$

Thus, the formula (3.10) follows from (3.11) and (3.12) when we use (3.4).

In view of the Corollary 2.1 and the equations (1.7) and (3.9) it follows from the above theorem that

$$(\mathfrak{M}_k \dot{\nabla}_h - \dot{\nabla}_h \mathfrak{M}_k)\, \dot{x}^i = 0\,.$$

Thus, we have the

COROLLARY 3.2. – *For the vector $\dot{x}^i$ the operators $\mathfrak{M}_k$ and $\dot{\nabla}_h$ are also commutative.*

The commutation formula for the unit vector l^i may be also derived from the above theorem. Using (2.9) and the relations $C^i_{hr}\, l^r = 0 = G^i_{khr}\, l^r$ we would have the

COROLLARY 3.3. – *For the unit vector l^i the operators $\mathfrak{M}_k$ and $\dot{\nabla}_h$ commute according to*

$$(3.12) \qquad (\mathfrak{M}_k \dot{\nabla}_h - \dot{\nabla}_h \mathfrak{M}_k)\, l^i = (-1/F)\, l^i\, C^r_{kh} \dot{\partial}_r\, (PF)\,.$$

THEOREM 3.3. – *For a vector-field $X^i(x, \dot{x})$ the operators $\mathfrak{M}_k$ and $\mathfrak{L}$ commute according to*

$$(3.14) \qquad (\mathfrak{L}\mathfrak{M}_k - \mathfrak{M}_k \mathfrak{L})\, X^i = X^j\, \mathfrak{L}\hat{G}^i_{jk} - (\dot{\partial}_j X^i)\, \mathfrak{L}\hat{G}^j_k\,.$$

Proof. – Since Lie differentiation satisfies the Leibnitz rule for the differentiation of a product we get from (2.7) that

$$(3.15)\qquad 2\,(\mathfrak{L}\mathfrak{M}_k - \mathfrak{L}\mathfrak{B}_k)\,X^i = (\mathfrak{L}\dot{\partial}_j X^i)\,\dot{\partial}_k\,(P\dot{x}^j) + (\dot{\partial}_j X^i)\,\mathfrak{L}\dot{\partial}_k\,(P\dot{x}^j) - $$
$$- (\mathfrak{L}X^j)\,\dot{\partial}_j\dot{\partial}_k\,(P\dot{x}^i) - X^j\,\mathfrak{L}\dot{\partial}_j\dot{\partial}_k\,(P\dot{x}^i)\,.$$

Next, applying the formula (2.7) for the vector X^i we have

$$(3.16)\qquad 2\,(\mathfrak{M}_k\,\mathfrak{L} - \mathfrak{B}_k\,\mathfrak{L})\,X^i = (\dot{\partial}_j\,\mathfrak{L}X^i)\,\dot{\partial}_k\,(P\dot{x}^j) - (\mathfrak{L}X^j)\,\dot{\partial}_j\dot{\partial}_k\,(P\dot{x}^i)\,.$$

As the Lie operator is commutative over the partial differential operator $\dot{\partial}_j$ [[5], eqn. (2.11)] we get, by subtracting (3.16) from (3.15) and using (3.3), that

$$(\mathfrak{L}\mathfrak{M}_k - \mathfrak{M}_k\,\mathfrak{L})\,X^i = X^j\,\mathfrak{L}\left\{G^i_{jk} - \frac{1}{2}\,\dot{\partial}_j\dot{\partial}_k\,(P\dot{x}^i)\right\} - (\dot{\partial}_j\,X^i)\,\mathfrak{L}\left\{G^j_k - \frac{1}{2}\,\dot{\partial}_k(P\dot{x}^j)\right\}.$$

Finally, using (2.1) and (2.4) this identity reduces to (3.14).

Noting $\mathfrak{L}\dot{x}^i = 0$ and (2.3) an important deduction of the above theorem may be had in the following form

COROLLARY 3.4. – *For the vector-field $\dot{x}^i$ the operators $\mathfrak{M}_k$ and $\mathfrak{L}$ are also commutative.*

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