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Massimo Furi, Alfonso Vignoli

On α-nonexpansive mappings and fixed points

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Analisi funzionale. — *On α–nonexpansive mappings and fixed points* (*). Nota di MASSIMO FURI e ALFONSO VIGNOLI presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — $T : D \to D$ una trasformazione continua definita in un sottoinsieme di uno spazio di Banach. Si danno delle condizioni sulla T affinché $\inf\{\|x - T(x)\| : x \in D\} = 0$ (Teor. 1, Teor. 2). Tali condizioni non assicurano l'esistenza di punti fissi; tuttavia, usando i risultati di precedenti Note (Teor. A, Teor. B) se ne deducono teoremi di punto fisso (Teor. 3, Teor. 4).

1. Let $T : X \to X$ be a mapping defined on a metric space (X, d). The mapping T has a fixed point if and only if the following two conditions are satisfied:

a) $\inf\{d(x, T(x)) : x \in X\} = 0$;

b) the real functional $I : x \mapsto d(x, T(x))$ has a minimum point.

In Section 2 we impose conditions on T in order that *a*) be satisfied. Furthermore, in Section 3, we give two fixed-point theorems. For this purpose we need some terminology.

Let $A \subset X$ be a bounded set of a metric space (X, d). By $\alpha(A)$ we denote the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε (see C. Kuratowski [1]). It is easily seen that $\alpha(A) = 0$ iff A is precompact, i.e. totally bounded.

Let $T : X \to X$ be a *continuous* mapping defined on a metric space (X, d) such that

$$\alpha(T(A)) \leq \lambda\alpha(A), \tag{1}$$

for any bounded subset $A \subset X$. If $\lambda < 1$ the mapping T is called *α–contractive* (see G. Darbo [2]). If $\lambda = 1$ then T is said to be *α–nonexpansive*. In the case when $\alpha(T(A)) < \alpha(A)$ for any bounded set $A \subset X$ such that $\alpha(A) > 0$, the mapping T is called *densifying* (see [3]).

Obviously if the mapping T is such that

$$d(T(x), T(y)) \leq \lambda d(x, y), \tag{2}$$

for all $x, y \in X$, then T satisfies condition (1).

It is well known that T is called *contractive* if (2) holds for $\lambda < 1$ and *nonexpansive* if (2) holds for $\lambda = 1$.

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Notice that contractive mappings and completely continuous mappings are densifying, as well as sums of these two types of mappings in Banach spaces.

Let $T : X \to X$ be a mapping defined on a metric space X and $F : X \times X \to R$ a lower semicontinuous real function. The mapping T is said to be *weakly* F*–contractive* if $F(T(x), T(y)) < F(x, y)$ for all $x, y \in X, x \neq y$.

The following two theorems will be useful for our purposes.

THEOREM A. (See [3]). *Let* $T : X \to X$ *be a densifying, weakly* F*–contractive mapping defined on a bounded complete metric space* X. *Then* T *has a unique fixed point in* X.

THEOREM B. (See [4]). *Let* $T : X \to X$ *be a densifying mapping defined on a bounded complete metric space* (X, d). *If* $\inf \{d(x, T(x)) : x \in X\} = 0$, *then* T *has at least one fixed point in* X.

2. Let $T : X \to X$ be a mapping defined on a metric space (X, d). Obviously, T satisfies condition *a*) if there exists a sequence of mappings $T_n : X \to X, n = 1, 2, \cdots$, such that $T_n(x)$ converges to $T(x)$ uniformly on Q and, for any $n \in N$, T_n has a fixed point $x_n \in X$. In [5] D. Göhde has shown that this is the case for nonexpansive mappings $T : Q \to Q$ defined on a bounded, closed and convex subset Q of a Banach space. The method used in [5] to prove condition *a*) is essentially the following:

Let $x_0 \in Q$ and, for any real number $\lambda \in [0, 1]$, define $T_\lambda : Q \to Q$ by $T_\lambda(x) = (1 - \lambda) x_0 + \lambda T(x)$. $T_\lambda(x)$ converges to $T(x)$ uniformly on Q as $\lambda \to 1$. Indeed,

$$\|T_\lambda(x) - T(x)\| = (1 - \lambda) \|x_0 - T(x)\| \leq (1 - \lambda)\, \delta(Q),$$

where $\delta(Q)$ is the diameter of Q. Moreover, for any $0 \leq \lambda < 1$, T_λ is a contractive mapping. Therefore, by the Banach contraction principle, there exists $x_\lambda \in Q$ such that $T_\lambda(x_\lambda) = x_\lambda$. But $\|x_\lambda - T(x_\lambda)\| = \|T_\lambda(x_\lambda) - T(x_\lambda)\|$ and, since $T_\lambda(x) \to T(x)$ uniformly on Q as $\lambda \to 1$, it follows that $\inf \{\|x - T(x)\| : x \in Q\} = 0$.

Using the same technique we shall prove the following two theorems.

THEOREM 1. *Let* $T : Q \to Q$ *be an* α*–nonexpansive mapping defined on a convex, closed and bounded subset* Q *of a Banach space* X. *Then* $\inf \{\|x - T(x)\| : x \in Q\} = 0$.

Proof. Let $x_0 \in Q$ be a given point and define $T_\lambda : Q \to Q$ by $T_\lambda(x) = (1 - \lambda) x_0 + \lambda T(x)$, $0 \leq \lambda < 1$. The mapping T_λ is α–contractive for any $0 \leq \lambda < 1$. Indeed, let $A \subset Q$; we have $T_\lambda(A) = (1 - \lambda) x_0 + \lambda T(A)$. Hence

$$\alpha(T_\lambda(A)) \leq \alpha((1 - \lambda) x_0) + \alpha(\lambda T(A)) = \lambda \alpha(T(A)).$$

Therefore, it follows, from a result of G. Darbo [2], that T_λ has at least a fixed point $x_\lambda \in Q$ for any $0 \leq \lambda < 1$.

Furthermore $T_\lambda(x)$ converges to $T(x)$ uniformly on Q as $\lambda \to 1$. But $\|x_\lambda - T(x_\lambda)\| = \|T_\lambda(x_\lambda) - T(x_\lambda)\|$, therefore $\|x_\lambda - T(x_\lambda)\| \to 0$ as $\lambda \to 1$. Then $\inf\{\|x - T(x)\| : x \in Q\} = 0$.

THEOREM 2. *Let* $T : S \to S$ *be an α–nonexpansive mapping on a star-shaped, closed and bounded subset* S *of a Banach space* X. *Let* $p : X \to [0, +\infty)$ *be a lower semicontinuous function such that*

α) $\quad x \neq 0 \Rightarrow p(x) > 0$,

β) $\quad p(tx) \leq tp(x) \quad$ *for* $\quad t \geq 0$.

Let T *satisfy the condition*

γ) $\quad p(T(x) - T(y)) \leq p(x - y), \qquad x, y \in S$.

Then $\inf\{\|x - T(x)\| : x \in S\} = 0$.

Proof. Let S be star-shaped with respect to x_0. Again, define $T_\lambda : S \to S$ by $T_\lambda(x) = (1 - \lambda)x_0 + \lambda T(x)$, $0 \leq \lambda < 1$. As in the proof of Theorem 1 the mapping T_λ is α–contractive (therefore, densifying) for any $0 \leq \lambda < 1$ and $T_\lambda(x)$ converges to $T(x)$ uniformly on S as $\lambda \to 1$. For $0 \leq \lambda < 1$ the mapping T_λ is weakly F–contractive, with $F(x, y) = p(x - y)$. Indeed, $F(T_\lambda(x), T_\lambda(y)) = p(T_\lambda(x) - T_\lambda(y)) = p(\lambda(T(x) - T(y))) \leq \lambda p(x - y) < F(x, y)$ for all $x, y \in S$, $x \neq y$. Therefore, from Theorem A it follows that T_λ has a unique fixed point $x_\lambda \in S$ for any $0 \leq \lambda < 1$. Then $\inf\{\|x - T(x)\| : x \in S\} = 0$.

Since nonexpansive mappings are also α–nonexpansive, the following example given in [6] shows that the assumptions of Theorem 1 (as well as those of Theorem 2) do not ensure the existence of a fixed point for the mapping T.

Example. Let c_0 be the Banach space of all real sequences which converge to zero, with the supremum norm. Let U be the unit ball of c_0. Define $T : U \to U$ as follows:

$$T : (\xi_1, \xi_2, \cdots) \mapsto (1, \xi_1, \xi_2, \cdots).$$

Obviously T is nonexpansive and does not have fixed points.

3. As a consequence of Theorem 1 and Theorem 2 we obtain the following two fixed point theorems.

THEOREM 3. *Let* $T : Q \to Q$ *be a densifying mapping defined on a closed, convex and bounded subset* Q *of a Banach space* X. *Then* T *has at least one fixed point.*

Proof. It follows immediately from Theorem B and Theorem 1.

THEOREM 4. *Let* $T : S \to S$ *be a densifying mapping defined on a bounded, closed and star-shaped subset of a Banach space* X. *Let* $p(T(x) - T(y)) \leq p(x - y)$, *where* p *is as in Theorem* 2. *Then* T *has at least one fixed point.*

Proof. It follows immediately from Theorem B and Theorem 2.

Remark. Theorem 3 is similar to a fixed-point theorem due to B. N. Sadovskij [7]. Namely, while in Theorem 3 the continuous mapping T satisfies the condition

i) " $\alpha(T(A)) < \alpha(A)$, $\forall A \subset Q$ such that $\alpha(A) > 0$",

in Sadovskij's result it satisfies

ii) " $\chi(T(A)) < \chi(A)$, $\forall A \subset Q$ such that $\chi(A) > 0$".

Here $\chi(A)$ denotes the infimum of all real numbers $\varepsilon > 0$ such that A admits a finite ε–net.

Observe that the number $\chi(A)$ has not all the properties of $\alpha(A)$. In particular $\chi(A)$ does not depend intrinsically on the bounded set A, as the following example shows.

Let A be an infinite orthonormal system of a Hilbert space H. We have $\chi(A) = 1$ if we consider A as a subset of H and $\chi(A) = \sqrt{2}$ if we consider A as a subset of itself.

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