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A remark on Schwarz Norm for operators

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Analisi matematica. — *A remark on Schwarz Norm for operators.* Nota di VASILE I. ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questo lavoro è data una nuova costruzione per le norme di Schwarz collegata con la teoria di dilatazione per operatori sugli spazi di Hilbert.

1. Let H be a complex Hilbert space and let $\mathfrak{L}(H)$ be the Banach algebra of all bounded operators on H. A norm $\| \|_1$ on $\mathfrak{L}(H)$, equivalent with the usual norm will be called a Schwarz Norm if the following assertion holds: if f is analytic and bounded by 1 in $\mathfrak{D} = \{z, |z| < 1\}$ with $f(0) = 0$ then $\|f(T)\|_1 \leq \|T\|_1$ when $\|T\|_1 < 1$.

In a recent paper J. Williams proposed a way to construct norms which have the above property.

In this Note we give an example of Schwarz Norm which is not in the class constructed by J. Williams. In § 3 we give a generalization of method of Williams, inspired from work in dilation theory.

2. If $T \in \mathfrak{L}(H)$, we denote by w_T its numerical radius and $\| \|$ the usual norm on $\mathfrak{L}(H)$. Our Schwarz Norm is

$$\|T\|_* = w_T + \|T\|.$$

LEMMA 1. $\| \|_*$ *is a Schwarz Norm.*

The fact that it is equivalent to usual norm is obvious. Also using von Neumann theorem on spectral sets for contractions and spectral mapping theorem one sees that $\| \|_*$ is a Schwarz norm.

LEMMA 2. $\| \|_*$ *is not in the class constructed in* [1].

The fact that $\| \|_*$ is not the usual norm is clear considering operators which are not normaloid. If we consider the operators, for $c > 0$

$$\frac{2}{c}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the norm of such operator in $\| \|_c$-norms of [1] we obtain that are different from $\| \|_*$ $(c = 2)$.

3. The classes of operators S_c, which play an important role in the definition of the norms $\| \|_c$, appear naturally in the study of unitary dilation of operators. H. Langer has given (see for example [2]) the following

(*) Nella seduta del 10 gennaio 1970.

generalization of the problem of unitary dilation: if A is a positive self-adjoint operator $0 < m\mathrm{I} \leq \mathrm{A} \leq \mathrm{MI}$ the class $\mathcal{C}_{\mathrm{A}}$ is the set of all operators whose powers admit a representation

$$\mathrm{QT}^n \mathrm{Q} = pr\, \mathrm{U}^n \qquad n = 1, 2, 3, \cdots$$

where $\mathrm{Q} = \mathrm{A}^{-1/2}$, U is a unitary operator in some Hilbert space $\mathrm{H}_1 \supset \mathrm{H}$. With this suggestion, we consider the class of operators generalising class S_c. This class is introduced by the definition 1.

Definition 1. If Q is a selfadjoint operator $0 < m\mathrm{I} \leq \mathrm{Q} \leq \mathrm{MI}$ the class S_{Q} is the set of all operators T with properties:

1) $\sigma(\mathrm{T}) \subset \{z, |z| \leq 1\}$
2) $\mathrm{Re}\left(1 + \sum_{n=1}^{\infty} \mathrm{QT}^n \mathrm{Q} z^n\right) \geq 0$.

Remark. The class of J. Williams is that for $\mathrm{Q} = c\mathrm{I}$ where c is a positive number.

Now, we shall give some properties of this class.

LEMMA 3. *If* $\mathrm{T} \in \mathrm{S}_{\mathrm{Q}}$ *then* $f(\mathrm{T}) \in \mathrm{S}_{\mathrm{Q}}$ *for each* $f \in \mathrm{R}_{\mathrm{D}}$ *with* $f(0) = 0$ *and* $\|f\|_{\mathrm{D}} \leq 1$.

Proof. Since $\mathrm{T} \in \mathrm{S}_{\mathrm{Q}}$ then there exists a positive measure μ_x on $[0, 2\pi]$ such that

$$\|x\|^2 + \sum_{n=1}^{\infty} z^n \langle \mathrm{QT}^n \mathrm{Q}x, x\rangle = \int \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}}\, d\mu_x(\theta)$$

if $|z| < 1$. We obtain thus

$$\langle \mathrm{QT}^n \mathrm{Q}x, x\rangle = 2 \int e^{in\theta}\, d\mu_x(\theta)$$

for all $n \geq 1$. Now, if f is a polynomial, we have

$$\langle \mathrm{Q}f(\mathrm{T})\,\mathrm{Q}x, x\rangle = 2 \int f(e^{i\theta})\, d\mu_x(\theta)$$

and for $[f(\mathrm{T})]^n = f(\mathrm{T})^n$

$$\langle \mathrm{Q}f(\mathrm{T})^n \mathrm{Q}x, x\rangle = 2 \int f^n(e^{i\theta})\, d\mu_x(\theta).$$

Since Q is a positive operator we conclude that $f^n(\mathrm{T})$ is bounded and thus

$$\left\langle \left(1 + \sum_1^{\infty} z^n \mathrm{Q}f^n(\mathrm{T})\,\mathrm{Q}\right) x, x \right\rangle = \|x\|^2 + 2 \sum_1^{\infty} z^n \int f^n(e^{i\theta})\, d\mu_x(\theta)$$

which implies that $f(\mathrm{T}) \in \mathrm{S}_{\mathrm{Q}}$. An approximation argument completes the proof.

For the following theorem, we need another equivalent form for the definition of the class S_Q. Is clear from the following

Definition 2. $T \in S_Q$ if and only if

1) $\sigma(T) \subset \{z, |z| \leq 1\}$;

2) $\operatorname{Re}\langle Q(1 - zT)^{-1} Qx, x\rangle - \langle Qx, Qx\rangle + \|x\|^2 \geq 0$;

for all $z, |z| < 1$, or

1′) $\sigma(T) \subset \{z, |z| \leq 1\}$;

2′) $\operatorname{Re}\langle Q(1 - zT)^{-1} Qx, x\rangle \geq \|Qx\|^2 - \|x\|^2 = \langle (Q^2 - 1)x, x\rangle$.

LEMMA 4. *The following assertions about* S_Q *classes are true*

1) $S_Q^* = S_Q$;

2) *if* $Q_1 \leq Q_2$ *then* $S_{Q_2} \subseteq S_{Q_1}$;

3) *if* $Q \geq 1$, $T \in S_Q$ *if and only if*

$$\|Q^{-2}y\|^2 \geq |\langle (1 - 2Q^{-1})Q^{-1}y, TQ^{-1}y\rangle| + \|(1 - Q^{-2})^{1/2}Q^{-1}y\|^2$$

4) *if* $Q \geq I$, S_Q *is convex.*

Proof. The assertion 1) is obvious. The assertion 2) may be proved as in [2] and we omit it.

From 2′) we have

$$\operatorname{Re}\langle y, Q^{-1}(1 - zT)Q^{-1}y\rangle \geq$$

$$\geq \langle (Q^2 - I)Q^{-1}(1 - zT)Q^{-1}y, Q^{-1}(1 - zT)Q^{-1}y\rangle\rangle =$$

$$= \langle (1 - Q^{-2})(1 - zT)Q^{-1}y, (1 - zT)Q^{-1}y\rangle$$

and thus

$$\|Q^{-1}y\|^2 \geq \operatorname{Re}\langle y, zQ^{-1}TQ^{-1}y\rangle + \langle (1 - Q^{-2})Q^{-1}y, Q^{-1}y\rangle -$$

$$- 2\operatorname{Re}\langle (1 - Q^{-2})Q^{-1}y, zTQ^{-1}y\rangle + \|(1 - Q^{-2})^{1/2} zTQ^{-1}y\|^2$$

which leads to

$$\|Q^{-1}y\|^2 \geq |\langle (1 - 2Q^{-2})y, Q^{-1}TQ^{-1}y\rangle| + \|Q^{-1}y\|^2 - \|Q^{-2}y\|^2 +$$

$$+ \|(1\ Q^{-2})^{1/2}TQ^{-1}y\|^2$$

and

$$\|Q^{-1}y\|^2 \geq |\langle (1 - 2Q^{-2})y, Q^{-1}TQy\rangle| + \|(1 - Q^{-2})^{1/2}TQ^{-1}y\|^2$$

which is equivalent to 3).

The assertion 4) follows from 3) using the parallelogram law.

We summarize the results in this section as

THEOREM. *If* Q *is a selfadjoint operator* $Q \geq I$ *then*

$$\| T \|_Q = \inf \{\lambda , T \in \lambda S_Q\}$$

is a Schwarz norm.

It is clear that lemmas 3 and 4 imply this theorem.

REFERENCES.

[1] J. P. WILLIAMS, *Schwarz Norms for operators*, « Pacif. J. Math. », *24* (1), 181–188 (1968).

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[3] V. ISTRĂŢESCU, *Remark on Schwarz norms for operators*, « Revue Roum. Math. Pures App. », *3*, 359–360 (1969).