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On a non-linear mixed problem for the Navier-Stokes equations. Nota II

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Analisi Matematica. — *On a non-linear mixed problem for the Navier-Stokes equations.* Nota II di GIOVANNI PROUSE (*), presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dimostra un teorema di esistenza della soluzione del problema posto nella Nota I.

3. Let Ω be the rectangle $0 < x_1 < l$, $0 < x_2 < k$ and Q the cylinder $\Omega \times 0 \vdash\!\dashv T$ $(T > 0.)$. *We shall prove*, in this §, *an existence theorem, for sufficiently small t, of a solution of the system of Navier-Stokes* (in a sense we shall specify later) *satisfying the initial condition*

$$\vec{u}(x_1, x_2, 0) = \vec{z}(x_1, x_2) \tag{3.1}$$

and the boundary conditions

$$\frac{1}{2} u_1^2(0, x_2, t) + p(0, x_2, t) = \alpha_1(x_2, t), \tag{3.2}$$

$$\frac{1}{2} u_1^2(l, x_2, t) + p(l, x_2, t) = \alpha_2(x_2, t), \tag{3.3}$$

$$p(x_1, 0, t) = -\beta_1(x_1, t)\, u_2(x_1, 0, t)\, |u_2(x_1, 0, t)|, \tag{3.4}$$

$$p(x_1, k, t) = \beta_2(x_1, t)\, u_2(x_1, k, t)\, |u_2(x_1, k, t)|, \tag{3.5}$$

$$u_1(x_1, 0, t) = u_1(x_1, k, t) = u_2(0, x_2, t) = u_2(l, x_2, t) = 0, \tag{3.6}$$

where α_1 and α_2 are given functions and β_1, β_2 are permeability coefficients, ≥ 0.

Let $\mathfrak{N}$ denote the manifold of vectors $\vec{v}(x_1, x_2) = \{v_1(x_1, x_2), v_2(x_1, x_2)\}$ indefinitely differentiable in Ω, with null divergence and such that $v_1(x_1, 0) = v_1(x_1, k) = v_2(0, x_2) = v_2(l, x_2) = 0$ and let N^σ be the closure of $\mathfrak{N}$ in $H^\sigma(\Omega)$. It is, evidently,

$$N^\sigma \subset H^\sigma(\Omega) \quad \text{when} \quad 0 \leq \sigma \leq 1; \tag{3.7}$$

moreover, the embedding of N^1 in N^0 is compact. There exists then, as is well known, a sequence of real positive non-decreasing numbers, $\{\lambda_j\}$, with $\lim_{j \to \infty} \lambda_j = +\infty$ and a sequence of elements $\{\vec{g}_j\} \in N^1$ such that $\{\vec{g}_j\}$ is a basis in N^0 and N^1 and

$$(\vec{g}_j, \vec{v})_{N^1} = \lambda_j (\vec{g}_j, \vec{v})_{N^0}, \quad \forall \vec{v} \in N^1 \quad , \quad (\vec{g}_j, \vec{g}_k)_{N^0} = \delta_{jk}. \tag{3.8}$$

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Let us denote, $\forall \sigma \geq 0$, by l_σ^2 the space of sequences $\{\alpha_j\}$ of real numbers such that

$$\sum_{=1}^{\infty} \lambda_j^\sigma \alpha_j^2 < +\infty. \tag{3.9}$$

Setting

$$(\{\alpha_j'\}, \{\alpha_j''\})_{l_\sigma^2} = \sum_{j=1}^{\infty} \lambda_j^\sigma \alpha_j' \alpha_j'', \tag{3.10}$$

l_σ^2 is a Hilbert space.

Finally, we denote by V_σ the space

$$\{\vec{v} \in N^0 ; \{(\vec{v}, g_j)_{N^0} \in l_\sigma^2\}, (\vec{v}, \vec{w})_{V_\sigma} = (\{(\vec{v}, g_j)_{N^0}\}, \{(\vec{w}, g_j)_{N^0}\})_{l_\sigma^2}. \tag{3.11}$$

It is evident that the spaces V_σ and l_σ^2 so defined are isomorphic and that N^0 and N^1 are isomorphic to V_0 and V_1 respectively.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, by the notation $[\mathcal{H}_1, \mathcal{H}_2]_\vartheta$ we shall denote the (Hilbert) space $\mathcal{H}_1^{1-\vartheta} \mathcal{H}_2^\vartheta$ which is intermediate between $\mathcal{H}_1$ and $\mathcal{H}_2$ according the definition given by Lions in [7] (1). We have (Baiocchi [8])

$$[V_\alpha, V_\beta]_\vartheta = V_{\alpha(1-\vartheta)+\beta\vartheta} \qquad (0 < \vartheta < 1) \tag{3.12}$$

and also (see for example Magenes [9])

$$H^\sigma(\Omega) = [L^2(\Omega), H^1(\Omega)]_\sigma \qquad (0 < \sigma < 1). \tag{3.13}$$

Hence, bearing in mind (3.7),

$$V_\sigma \subseteq H^\sigma(\Omega) \qquad (0 \leq \sigma \leq 1). \tag{3.14}$$

Let us now give the definition of *solution* of system (1.1).

Setting $b(\vec{u}, \vec{v}, \vec{w}) = \int_\Omega \sum_{j,k=1}^{2} u_j \frac{\partial v_k}{\partial x_j} w_k \, d\Omega$, *we shall say that* $\vec{u}(t)$ *is a weak solution in* $0 \vdash\!\dashv \bar{t}$ *of system* (1.1), *satisfying the boundary conditions* (3.2), (3.3), (3.4), (3.5), (3.6) *if* $\vec{u}(t) \in L^\infty(0 \vdash\!\dashv \bar{t}; V_0) \cap L^2(0 \vdash\!\dashv \bar{t}; V_1)$ *and satisfies the equation*

$$\begin{aligned}
\int_0^{\bar{t}} \{ &-(\vec{u}(t), \vec{h}'(t))_{V_0} + \mu(\vec{u}(t), \vec{h}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{h}(t)) - \\
&- (\vec{f}(t), \vec{h}(t))_{V_0}\} \, dt = \\
= \int_0^{\bar{t}} \int_0^k &\Big\{ \Big(\alpha_1(x_2, t) - \frac{1}{2} u_1^2(0, x_2, t)\Big) h_1(0, x_2, t) - \\
&- \Big(\alpha_2(x_2, t) - \frac{1}{2} u_1^2(l, x_2, t)\Big) h_1(l, x_2, t) \Big\} \, dx_2 \, dt -
\end{aligned} \tag{3.15}$$

(1) Per tutte le citazioni bibliografiche vedi Nota I, questi « Rendiconti », vol. XLVIII, fasc. 1, p. 26 (1970).

$$-\int_0^{\bar{t}}\int_0^{l} \{\beta_1(x_1,t)\,|u_2(x_1,0,t)|\,u_2(x_1,0,t)\,h_2(x_1,0,t) + \beta_2(x_1,t)\,|u_2(x_1,k,t)|\,u_2(x_1,k,t)\,h_2(x_1,k,t)\}\,dx_1\,dt$$

$\forall \vec{h}(t) \in C^0(0 \vdash\!\dashv \bar{t}\,;V_1)$ with $\vec{h}'(t) \in L^1(0 \vdash\!\dashv \bar{t}\,;V_0)$, $\vec{h}(\bar{t}) = \vec{h}(0) = 0$.

Relation (3.15) can be obtained in an obvious way multiplying the first of (1.2) by $\vec{h}(t)$ and bearing in mind the second of (1.2) and the boundary conditions.

Let us now prove the following existence theorem.

Assume that:

I) $\vec{f} \in L^2(Q)$ $\qquad (Q = \Omega \times 0 \vdash\!\dashv T)$;

II) $\alpha_1(t) = \{\alpha_1(x_2,t)\,;x_2 \in \Gamma_1\} \in L^\infty(0 \vdash\!\dashv T\,;L^2(\Gamma_1))$ $\qquad (\Gamma_1 = \{x_1 = 0\,;x_2 \in 0 \vdash\!\dashv k\})$;

$\alpha_2(t) = \{\alpha_2(x_2,t)\,;x_2 \in \Gamma_3\} \in L^\infty(0 \vdash\!\dashv T\,;L^2(\Gamma_3))$ $\qquad (\Gamma_3 = \{x_1 = l\,;x_2 \in 0 \vdash\!\dashv k\})$;

III) $\beta_1(t) = \{\beta_1(x_1,t)\,;x_1 \in \Gamma_2\} \in L^\infty(0 \vdash\!\dashv T\,;L^\infty(\Gamma_2))$ $\qquad (\Gamma_2 = \{x_1 \in 0 \vdash\!\dashv l\,;x_2 = 0\})$;

$\beta_2(t) = \{\beta_2(x_1,t)\,;x_1 \in \Gamma_4\} \in L^\infty(0 \vdash\!\dashv T\,;L^\infty(\Gamma_4))$ $\qquad (\Gamma_4 = \{x_1 \in 0 \vdash\!\dashv l\,;x_2 = k\})$;

$\beta_j(x_1,t) \geq 0 \qquad (j = 1,2)$;

IV) $\vec{z} \in V_\sigma$, $\quad 0 < \sigma < \frac{1}{2}$.

It is then possible to find $\bar{t}$, *with* $0 < \bar{t} \leq T$, *such that in* $0 \vdash\!\dashv \bar{t}$ *there exists a solution of the system of Navier-Stokes satisfying the initial condition* (3.1) *and the boundary conditions* (3.2), (3.3), (3.4), (3.5), (3.6). *Moreover,*

$$\vec{u}(t) \in L^\infty(0 \vdash\!\dashv \bar{t}\,;V_\sigma) \cap L^2(0 \vdash\!\dashv \bar{t}\,;V_{\sigma+1})\,. \tag{3.16}$$

Bearing in mind the properties of the sequence $\{\vec{g}_j\}$, let us denote by $\vec{z}_n$ a linear combination of $\vec{g}_1, \vec{g}_2, \cdots, \vec{g}_n$ such that

$$\lim_{n\to\infty} \vec{z}_n \underset{V_\sigma}{=} \vec{z}\,. \tag{3.17}$$

Setting then

$$\vec{u}_n(t) = \sum_{j=1}^{n} \alpha_{jn}(t)\,\vec{g}_j\,, \tag{3.18}$$

let us consider, for the system of n ordinary differential equations

$$(3.19)\qquad (\vec{u}_n'(t), \vec{g}_j)_{V_0} + \mu(\vec{u}_n(t), \vec{g}_j)_{V_1} + b(\vec{u}_n(t), \vec{u}_n(t), \vec{g}_j) - (\vec{f}(t), \vec{g}_j)_{V_0} -$$

$$-\int_0^k \left\{ \left(\alpha_1(x_2, t) - \frac{1}{2} u_1^2(0, x_2, t)\right) g_{1,j}(0, x_2) - \right.$$

$$\left. - \left(\alpha_2(x_2, t) - \frac{1}{2} u_1^2(l, x_2, t)\right) g_{1,j}(l, x_2) \right\} \mathrm{d}x_2 +$$

$$+ \int_0^l \{ \beta_1(x_1, t) \, | u_{2,n}(x_1, 0, t) | \, u_{2,n}(x_1, 0, t) \, g_{2,j}(x_1, 0) +$$

$$+ \beta_2(x_1, t) \, | u_{2,n}(x_1, k, t) | \, u_{2,n}(x_1, k, t) \, g_{2,j}(x_1, k) \} \, \mathrm{d}x_1 = 0$$

the initial value problem

$$(3.20)\qquad \vec{u}_n(0) = \vec{z}_n .$$

As can easily be seen, the assumptions of the existence and uniqueness theorem for the solution of the initial value problem for systems of ordinary differential equations are verified. We shall therefore denote by $\vec{u}_n(t)$ the solution of (3.19), (3.20) which exists, for $t > 0$, in an appropriately small neighbourhood of the origin.

Let us now establish, for $\vec{u}_n(t)$, some evaluations which are independent of n.

Multiplying (3.19), by $\lambda_j^\sigma \alpha_{jn}(t)$ and adding, we obtain, setting $\sum_{j=1}^{n} \lambda_j^\sigma \alpha_{jn}(t) \vec{g}_j = A^\sigma \vec{u}_n(t)$ and bearing in mind (3.10) and (3.11),

$$(3.21)\qquad \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \vec{u}_n(t) \|_{V_\sigma}^2 + \mu \| \vec{u}_n(t) \|_{V_{\sigma+1}}^2 + b(\vec{u}_n(t), \vec{u}_n(t), A^\sigma \vec{u}_n(t)) =$$

$$= (\vec{f}(t), A^\sigma \vec{u}_n(t))_{V_0} + \int_0^k \left\{ \left(\alpha_1(x_2, t) - \frac{1}{2} u_{1,n}^2(0, x_2, t)\right) A^\sigma u_{1,n}(0, x_2, t) - \right.$$

$$\left. - \left(\alpha_2(x_2, t) - \frac{1}{2} u_{1,n}^2(l, x_2, t)\right) A^\sigma u_{1,n}(l, x_2, t) \right\} \mathrm{d}x_2 -$$

$$- \int_0^l \{ \beta_1(x_1, t) \, | u_{2,n}(x_1, 0, t) | \, u_{2,n}(x_1, 0, t) \, A^\sigma u_{2,n}(x_1, 0, t) +$$

$$+ \beta_2(x_1, t) \, | u_{2,n}(x_1, k, t) | \, u_{2,n}(x_1, k, t) \, A^\sigma u_{2,n}(x_1, k, t) \} \, \mathrm{d}x_1 .$$

Hence, denoting by γ_0 the "trace" operator on the boundary Γ and by $\vec{u}_n^2$ the vector with components $u_{1,n}^2$ and $u_{2,n}^2$, we obtain, bearing in mind assumption III),

$$(3.22)\qquad \frac{1}{2}\frac{d}{dt}\|\vec{u}_n(t)\|^2_{V_\sigma}+\mu\|\vec{u}_n(t)\|^2_{V_{\sigma+1}}\leq |b(\vec{u}_n(t),\vec{u}_n(t),A^\sigma\vec{u}_n(t))|+$$
$$+|(\vec{f}(t),A^\sigma\vec{u}_n(t))_{V_0}|+c_1|(\gamma_0\vec{u}^2_n(t),\gamma_0 A^\sigma\vec{u}_n(t))_{L^2(\Gamma)}|+$$
$$+|(\alpha_1(t),\gamma_0 A^\sigma u_{1,n}(t))_{L^2(\Gamma_1)}|+|(\alpha_2(t),\gamma_0 A^\sigma u_{1,n}(t))_{L^2(\Gamma_3)}|\leq$$
$$\leq|b(\vec{u}_n(t),\vec{u}_n(t),A^\sigma\vec{u}_n(t))|+\|\vec{f}(t)\|_{V_0}\|A^\sigma\vec{u}_n(t)\|_{V_0}+c_1\|\gamma_0\vec{u}^2_n(t)\|_{L^2(\Gamma)}\cdot$$
$$\cdot\|\gamma_0 A^\sigma\vec{u}_n(t)\|_{L^2(\Gamma)}+(\|\alpha_1(t)\|_{L^2(\Gamma_1)}+\|\alpha_2(t)\|_{L^2(\Gamma_3)})\|\gamma_0 A^\sigma\vec{u}_n(t)\|_{L^2(\Gamma)}\,.$$

On the other hand, by well known embedding and isomorphism theorems (cfr. Magenes [9], Nikolskii [10]) we have, $\forall\varepsilon$ with $0<\varepsilon<\frac{1}{2}-\sigma$,

$$(3.23)\qquad \|\gamma_0 A^\sigma\vec{u}_n(t)\|_{L^2(\Gamma)}\leq c_2\|A^\sigma\vec{u}_n(t)\|_{H^{(1/2)+\varepsilon}(\Omega)}\leq c_3\|A^\sigma\vec{u}_n(t)\|_{V_{(1/2)+\varepsilon}}=$$
$$=c_3\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}}\,,$$

$$(3.24)\qquad \|\gamma_0\vec{u}^2_n(t)\|_{L^2(\Gamma)}=\|\gamma_0\vec{u}_n(t)\|^2_{L^4(\Gamma)}\leq c_4\|\vec{u}_n(t)\|^2_{H^{3/4}(\Omega)}\leq c_5\|\vec{u}_n(t)\|^2_{V_{3/4}}\,,$$

$$(3.25)\qquad |b(\vec{u}_n(t),\vec{u}_n(t),A^\sigma\vec{u}_n(t))|\leq\|\vec{u}_n(t)\|_{L^4(\Omega)}\|\vec{u}_n(t)\|_{H^1(\Omega)}\|A^\sigma\vec{u}_n(t)\|_{L^4(\Omega)}\leq$$
$$\leq c_6\|\vec{u}_n(t)\|_{H^{1/2}(\Omega)}\|\vec{u}_n(t)\|_{H^1(\Omega)}\|A^\sigma\vec{u}_n(t)\|_{H^{1/2}(\Omega)}\leq$$
$$\leq c_7\|\vec{u}_n(t)\|_{V_{1/2}}\|\vec{u}_n(t)\|_{V_1}\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)}}\,.$$

It follows therefore, from (3.22), (3.23), (3.24), (3.25), by the hypotheses made, that

$$(3.26)\quad \frac{1}{2}\frac{d}{dt}\|\vec{u}_n(t)\|^2_{V_\sigma}+\mu\|\vec{u}_n(t)\|^2_{V_{\sigma+1}}\leq c_7\|\vec{u}_n(t)\|_{V_{1/2}}\|\vec{u}_n(t)\|_{V_1}\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)}}+$$
$$+c_8\|\vec{u}_n(t)\|^2_{V_{3/4}}\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}}+\chi(t)\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}}$$

where $\chi(t)\in L^2(0\,\text{|—|}\,T)$, $\chi(t)\geq 0$.

Moreover, by theorems on interpolation spaces (cfr. Baiocchi [8]) we obtain

$$(3.27)\qquad\begin{aligned}&\|\vec{u}_n(t)\|_{V_{3/4}}\leq c_9\|\vec{u}_n(t)\|^{1/2}_{V_{1/2}}\|\vec{u}_n(t)\|^{1/2}_{V_1}\,,\\&\|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}}\leq c_{10}\|\vec{u}_n(t)\|^{(1/2)-\sigma-\varepsilon}_{V_\sigma}\|\vec{u}_n(t)\|^{(1/2)+\sigma+\varepsilon}_{V_{\sigma+1}}\,,\\&\|\vec{u}_n(t)\|_{V_{1/2}}\leq c_{11}\|u_n(t)\|^{(1/2)+\sigma}_{V_\sigma}\|\vec{u}_n(t)\|^{(1/2)-\sigma}_{V_{\sigma+1}}\,,\\&\|\vec{u}_n(t)\|_{V_1}\leq c_{12}\|\vec{u}_n(t)\|^{\sigma}_{V_\sigma}\|\vec{u}_n(t)\|^{1-\sigma}_{V_{\sigma+1}}\,.\end{aligned}$$

From (3.27) it follows then that

$$(3.28)\quad c_7 \|\vec{u}_n(t)\|_{V_{1/2}} \|\vec{u}_n(t)\|_{V_1} \|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)}} + c_8 \|\vec{u}_n(t)\|^2_{V_{3/4}} \|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}} \le$$

$$\le c_{13} \|\vec{u}_n(t)\|_{V_{1/2}} \|\vec{u}_n(t)\|_{V_1} \|\vec{u}_n(t)\|_{V_{2\sigma+(1/2)+\varepsilon}} \le c_{14} \|\vec{u}_n(t)\|^{1+\sigma+\varepsilon}_{V_\sigma} \|\vec{u}_n(t)\|^{2-\sigma+\varepsilon}_{V_{\sigma+1}} \le$$

$$\le c_{14} \left(\nu \|\vec{u}_n(t)\|^2_{V_{\sigma+1}} + \nu^{\frac{2-\sigma+\varepsilon}{\sigma-\varepsilon}} \|\vec{u}_n(t)\|_{V_\sigma}^{2\frac{1+\sigma-\varepsilon}{\sigma-\varepsilon}}\right).$$

Choosing $\nu = \frac{\mu}{2c_{14}}$ and introducing (3.28) in (3.26) we obtain

$$(3.29)\quad \frac{1}{2}\frac{d}{dt}\|\vec{u}_n(t)\|^2_{V_\sigma} + \frac{\mu}{2}\|\vec{u}_n(t)\|^2_{V_{\sigma+1}} \le c_{15}\|\vec{u}_n(t)\|_{V_\sigma}^{2\frac{1+\sigma-\varepsilon}{2-\sigma+\varepsilon}} + c_{16}\,\chi(t)\|\vec{u}_n(t)\|_{V_{\sigma+1}}.$$

Hence there exists $\bar{t}$, with $0 < \bar{t} \le T$ such that

$$(3.30)\qquad \sup_{0\le t\le \bar{t}} \|\vec{u}_n(t)\|_{V_\sigma} \le M_1 \quad , \quad \int_0^{\bar{t}} \|\vec{u}_n(t)\|^2_{V_{\sigma+1}}\, dt \le M_2 ,$$

where $\bar{t}$, M_1, M_2, are independent of n.

It is therefore possible to select from $\{\vec{u}_n(t)\}$ a subsequence (again denoted by $\{\vec{u}_n(t)\}$) such that

$$(3.31)\qquad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^2(0 \vdash\!\dashv \bar{t};\, V_{\sigma+1})}{=} \vec{u}(t) \quad , \quad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^\infty(0\vdash\!\dashv \bar{t};\, V_\sigma)}{=} \vec{u}(t) ,$$

respectively in the weak and weak* topologies.

Moreover, by the same theorems utilized above we obtain, since $\varepsilon < \frac{1}{2} - \sigma$,

$$(3.32)\qquad |(\vec{w}, \vec{v})_{V_1}| \le c_{17} \|\vec{w}\|_{V_{\sigma+1}} \|\vec{v}\|_{V_{1-\sigma}}$$

$$(3.33)\qquad |b(\vec{w}, \vec{w}, \vec{v})| \le c_{18} \|\vec{w}\|_{L^{2/(1-\sigma)}(\Omega)} \|\vec{w}\|_{H^1(\Omega)} \|\vec{v}\|_{L^{2/\sigma}(\Omega)} \le$$

$$\le c_{19} \|\vec{w}\|_{H^\sigma(\Omega)} \|\vec{w}\|_{H^1(\Omega)} \|\vec{v}\|_{H^{1-\sigma}(\Omega)} \le c_{20} \|\vec{w}\|_{V_\sigma} \|\vec{w}\|_{V_1} \|\vec{v}\|_{V_{1-\sigma}} ,$$

$$(3.34)\quad \Bigg| \int_0^k \Big\{\Big(\alpha_1(x_2, t) - \frac{1}{2} w_1^2(0, x_2)\Big) v_1(0, x_2) - \Big(\alpha_2(x_2, t) - \frac{1}{2} w_1^2(l, x_2)\Big) v_1(l, x_2)\Big\}\, dx_2 +$$

$$+ \int_0^l \{\beta_1(x_1, t)\, |w_2(x_1, 0)|\, w_2(x_1, 0)\, v_2(x_1, 0) + \beta_2(x_1, t)\, |w_2(x_1, k)| \cdot$$

$$w_2(x_1, k)\, v_2(x_1, k)\}\, dx_1 \Bigg| \le$$

$$\le c_{21}(t) \left(\|\vec{w}\|^2_{V_{3/4}} + c_{22}(t)\right) \|\vec{v}\|_{H^{(1/2)+\varepsilon}(\Omega)} \le c_{23}(t) \left(\|\vec{w}\|^2_{V_{3/4}} + c_{22}(t)\right) \|\vec{v}\|_{V_{1-\sigma}} ,$$

being, by the assumptions II), III) $c_{21}(t)$, $c_{22}(t)$, $c_{23}(t) \in L^\infty(0 \vdash\!\dashv T)$.

Let $\vec{v} = \sum_{j=1}^{\infty} \psi_j \vec{g}_j$ be an arbitrary element of V_1 and set $\vec{v}_n = \sum_{j=1}^{n} \psi_j \vec{g}_j$; multiplying (3.19) by ψ_j and adding, we obtain

$$(3.35) \qquad (\vec{u}_n'(t), \vec{v}_n)_{V_0} + \mu (\vec{u}_n(t), \vec{v}_n)_{V_1} + b(\vec{u}_n(t), \vec{u}_n(t), \vec{v}_n) - (\vec{f}(t), \vec{v}_n)_{V_0} -$$
$$- \int_0^k \Big\{ \Big(\alpha_1(x_2, t) - \frac{1}{2} u_{1,n}^2(0, x_2, t) \Big) v_{1,n}(0, x_2) -$$
$$- \Big(\alpha_2(x_2, t) - \frac{1}{2} u_{1,n}^2(l, x_2, t) \Big) v_{1,n}(l, x_2) \Big\} dx_2 +$$
$$+ \int_0^l \{ \beta_1(x_1, t) \, | u_{2,n}(x_1, 0, t) | \, u_{2,n}(x_1, 0, t) \, v_{2,n}(x_1, 0) +$$
$$+ \beta_2(x_1, t) \, | u_{2,n}(x_1, k, t) | \, u_{2,n}(x_1, k, t) \, v_{2,n}(x_1, k) \} \, dx_1 = 0.$$

Therefore, by (3.32), (3.33), (3.34), (3.35), denoting by $\langle \cdot , \cdot \rangle$ the duality between $V_{\sigma-1}$ and $V_{1-\sigma}$,

$$| \langle \vec{u}_n'(t), \vec{v}_n \rangle | = | (\vec{u}_n'(t), \vec{v}_n)_{V_0} | \leq \mu c_{17} \| \vec{u}_n(t) \|_{V_{\sigma+1}} \| \vec{v}_n \|_{V_{1-\sigma}} +$$
$$+ c_{20} \| \vec{u}_n(t) \|_{V_\sigma} \| \vec{u}_n(t) \|_{V_1} \| \vec{v}_n \|_{V_{1-\sigma}} + c_{23}(t) \left(\| \vec{u}_n(t) \|_{V_{3/4}}^2 + c_{22}(t) \right) \| \vec{v}_n \|_{V_{1-\sigma}} \leq$$
$$\leq \left(\mu c_{17} \| \vec{u}_n(t) \|_{V_{\sigma+1}} + c_{24} \| \vec{u}_n(t) \|_{V_\sigma} \| \vec{u}_n(t) \|_{V_{\sigma+1}} + \right.$$
$$\left. + c_{25}(t) \| \vec{u}_n(t) \|_{V_{\sigma+1}}^{(3/2)-2\sigma} \| \vec{u}_n(t) \|_{V_\sigma}^{(1/2)+2\sigma} + c_{22}(t) \right) \| \vec{v}_n \|_{V_{1-\sigma}} .$$

Hence, by (3.30),

$$(3.36) \qquad | \langle \vec{u}_n'(t), \vec{v}_n \rangle | \leq c_{26}(t) \| \vec{v}_n \|_{V_{1-\sigma}} ,$$

with $c_{26}(t) \in L^{4/3}(0 \,|\!-\!|\, \bar{t})$.

By (3.8) and the definition of the sequence $\{\vec{g}_j\}$, we have, on the other hand,

$$(\vec{u}_n'(t), \vec{v}_n)_{V_\sigma} = (\vec{u}_n'(t), \vec{v})_{V_\sigma} \qquad \| \vec{v}_n \|_{V_{1-\sigma}} \leq \| \vec{v} \|_{V_{1-\sigma}} .$$

Consequently, (3.36) becomes, $\forall \vec{v} \in V_1$,

$$| \langle \vec{u}_n'(t), \vec{v} \rangle | \leq c_{26}(t) \| \vec{v} \|_{V_{1-\sigma}} .$$

As V_1 is dense in $V_{1-\sigma}$, we obtain, finally,

$$(3.37) \qquad \| \vec{u}_n'(t) \|_{V_{\sigma-1}} \leq c_{26}(t) ,$$

being $c_{26}(t) \in L^{4/3}(0 \,|\!-\!|\, \bar{t})$ independent of n.

From (3.30), (3.37) it follows, by an embedding theorem, that

$$(3.38)\qquad \vec{u}_n(t) \in H^{1,\frac{4}{3}}(0\,|\!-\!|\,\bar{t}\,;V_{\sigma-1}) \cap L^2(0\,|\!-\!|\,\bar{t}\,;V_{\sigma+1}) \subset$$

$$\subset H^{3/4}(0\,|\!-\!|\,\bar{t}\,;V_{\sigma-1}) \cap L^2(0\,|\!-\!|\,\bar{t}\,;V_{\sigma+1}) \subset H^{3/16}(0\,|\!-\!|\,\bar{t}\,;V_{\sigma+(1/2)})$$

and, moreover, since the evaluations obtained are independent of n,

$$(3.39)\qquad \|\vec{u}_n(t)\|_{H^{3/16}(0|-|\bar{t}\,;V_{\sigma+(1/2)})} \le M_3\,,$$

with M_3 independent of n.

As the embedding of $H^{3/16}(0\,|\!-\!|\,\bar{t}\,;V_{\sigma+(1/2)})$ in $L^2(0\,|\!-\!|\,\bar{t}\,;L^4(\Omega))$ is compact, it is possible to select from $\{\vec{u}_n(t)\}$ a subsequence (again denoted by $\{\vec{u}_n(t)\}$) such that

$$(3.40)\qquad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^2(0|-|\bar{t}\,;L^4(\Omega))}{=} \vec{u}(t)\,.$$

On the other hand,

$$(3.41)\qquad \|\gamma_0\vec{u}_n(t)\|_{H^{3/16}(0|-|\bar{t}\,;H^\sigma(\Gamma))} \le c_{28}\|\vec{u}_n(t)\|_{H^{3/16}(0|-|\bar{t}\,;H^{\sigma+(1/2)}(\Omega))} \le$$

$$\le c_{29}\|\vec{u}_n(t)\|_{H^{3/16}(0|-|\bar{t}\,;V_{\sigma+(1/2)})} \le M_4\,.$$

Hence, as the embedding of $H^{3/16}(0\,|\!-\!|\,\bar{t}\,;H^\sigma(\Gamma))$ in $L^2(0\,|\!-\!|\,\bar{t}\,;L^2(\Gamma))$ is compact, we may assume that

$$(3.42)\qquad \lim_{n\to\infty}\gamma_0\vec{u}_n(t) \underset{L^2(0|-|\bar{t}\,;L^2(\Gamma))}{=} \gamma_0\vec{u}(t)\,.$$

Let

$$(3.43)\qquad \vec{h}(t) = \sum_{\text{finite}} \nu_j(t)\,\vec{g}_j \quad,\quad \nu_j(t)\in C^1(0\,|\!-\!|\,\bar{t}) \quad,\quad \nu_j(0)=\nu_j(\bar{t})=0\,.$$

We have then

$$(3.44)\qquad \lim_{n\to\infty}\int_0^{\bar{t}} (\vec{u}_n'(t)\,,\vec{h}(t))_{V_0}\,dt = -\lim_{n\to\infty}\int_0^{\bar{t}} (\vec{u}_n(t)\,,\vec{h}'(t))_{V_0}\,dt =$$

$$= -\int_0^{\bar{t}} (\vec{u}(t)\,,h'(t))_{V_0}\,dt\,,$$

$$(3.45)\qquad \lim_{n\to\infty}\int_0^{\bar{t}} (\vec{u}_n(t)\,,\vec{h}(t))_{V_1}\,dt = \int_0^{\bar{t}} (\vec{u}(t)\,,\vec{h}(t))_{V_1}\,dt\,,$$

$$\lim_{n\to\infty}\int_0^{\bar t} b(\vec{u}_n(t), \vec{u}_n(t), \vec{h}(t))\,dt = \int_0^{\bar t} b(\vec{u}(t), \vec{u}(t), \vec{h}(t))\,dt, \tag{3.46}$$

$$\lim_{n\to\infty}\Bigg[\int_0^{\bar t}\int_0^{k}\Big\{\Big(\alpha_1(x_2,t)-\frac{1}{2}u_{1,n}(0,x_2,t)\Big)h_1(0,x_2,t)- \tag{3.47}$$

$$-\Big(\alpha_2(x_2,t)-\frac{1}{2}u_{1,n}(l,x_2,t)\Big)h_1(l,x_2,t)\Big\}\,dx_2\,dt-$$

$$-\int_0^{\bar t}\int_0^{l}\{\beta_1(x_1,t)\,|u_{2,n}(x_1,0,t)|\,u_{2,n}(x_1,0,t)\,h_2(x_1,0,t)+$$

$$+\beta_2(x_1,t)\,|u_{2,n}(x_1,k,t)|\,u_{2,n}(x_1,k,t)\,h_2(x_1,k,t)\}\,dx_1\,dt\Bigg]=$$

$$=\int_0^{\bar t}\int_0^{k}\Big\{\Big(\alpha_1(x_2,t)-\frac{1}{2}u_1(0,x_2,t)\Big)h_1(0,x_2,t)-$$

$$-\Big(\alpha_2(x_2,t)-\frac{1}{2}u_1(l,x_2,t)\Big)h_1(l,x_2,t)\Big\}\,dx_2\,dt-$$

$$-\int_0^{\bar t}\int_0^{l}\{\beta_1(x_1,t))\,|u_2(x_1,0,t)|\,u_2(x_1,0,t)\,h_2(x_1,0,t)+$$

$$+\beta_2(x_1,t)\,|u_2(x_1,k,t)|\,u_2(x_1,k,t)\,h_2(x_1,k,t)\}\,dx_1\,dt.$$

In fact, (3.44), (3.45) follow directly from (3.31).

To prove (3.46), observe that, setting $\vec{w}_n = \vec{u}_n - \vec{u}$,

$$b(\vec{u}_n, \vec{u}_n, \vec{h}) - b(\vec{u}, \vec{u}, \vec{h}) = b(\vec{w}_n, \vec{u}_n, \vec{h}) - b(\vec{u}, \vec{w}_n, \vec{h}) =$$

$$= b(\vec{w}_n, \vec{u}_n, \vec{h}) + b(\vec{u}, \vec{h}, \vec{w}_n) - \int_\Gamma \sum_{j,k=1}^{2} u_j\, w_{k,n}\, h_k \cos\vec{n}x_j\, d\Gamma.$$

Hence, being $\|\gamma_0\vec{v}\|_{L^4(\Gamma)} \le c_{30}\|\gamma_0\vec{v}\|_{H^{1/4}(\Gamma)} \le c_{31}\|\vec{v}\|_{V_{3/4}}$, we obtain

$$|b(\vec{u}_n, \vec{u}_n, \vec{h}) - b(\vec{u}, \vec{u}, \vec{h})| \le c_{32}\|\vec{w}_n\|_{L^4(\Omega)}\|\vec{u}_n\|_{V_1}\|\vec{h}\|_{L^4(\Omega)} + \tag{3.48}$$

$$+ c_{32}\|\vec{u}\|_{L^4(\Omega)}\|\vec{h}\|_{V_1}\|\vec{w}_n\|_{L^4(\Omega)} + c_{33}\|\gamma_0\vec{w}_n\|_{L^2(\Gamma)}\|\vec{u}\|_{V_{3/4}}\|\vec{h}\|_{V_{3/4}}.$$

Relation (3.46) follows immediately from (3.30), (3.40), (3.42) (3.48).

Finally, observing that

$$|\vec{u}_n^2 - \vec{u}^2|^2 = (u_{1,n}^2 - u_1)^2 + (u_{2,n} - u_2)^2 = (u_{1,n} - u_1)^2 (u_{1,n} + u_1)^2 + \\ + (u_{2,n} - u_2)^2 (u_{2,n} + u_2)^2 \leq [(u_{1,n} - u_1)^2 + (u_{2,n} - u_2)^2] \cdot \\ \cdot [(u_{1,n} + u_1)^2 + (u_{2,n} + u_2)^2] = |\vec{u}_n - \vec{u}|^2 \, |\vec{u}_n + \vec{u}|^2 ,$$

$$\left| |\vec{u}_n| \, \vec{u}_n - |\vec{u}| \, \vec{u} \right| = \left| |\vec{u}_n| (\vec{u}_n - \vec{u}) + \vec{u} (|\vec{u}_n| - |\vec{u}|) \right| \leq \\ \leq |\vec{u}_n| \, |\vec{u}_n - \vec{u}| + |\vec{u}| \, |\vec{u}_n - \vec{u}| ,$$

we obtain, analogously to (3.48),

$$\left| \int_\Gamma (\vec{u}_n^2 - \vec{u}^2) \times \vec{h} \, d\Gamma \right| \leq \int_\Gamma |\vec{w}_n| \, |\vec{u}_n + \vec{u}| \, |\vec{h}| \, d\Gamma \leq \tag{3.49}$$

$$\leq \|\gamma_0 \vec{w}_n\|_{L^2(\Gamma)} \|\gamma_0 \vec{u}_n + \gamma_0 \vec{u}\|_{L^4(\Gamma)} \|\gamma_0 \vec{h}\|_{L^4(\Gamma)} \leq c_{34} \|\gamma_0 \vec{w}_n\|_{L^2(\Gamma)} \|\vec{u}_n + \vec{u}\|_{V_1} \|\vec{h}\|_{V_1} ,$$

$$\left| \int_\Gamma (|\vec{u}_n| \, \vec{u}_n - |\vec{u}| \, \vec{u}) \times \vec{h} \, d\Gamma \right| \leq \int_\Gamma |\vec{w}_n| (|\vec{u}_n| + |\vec{u}|) |\vec{h}| \, d\Gamma \leq$$

$$\leq c_{35} \|\gamma_0 \vec{w}_n\|_{L^2(\Gamma)} (\|\vec{u}_n\|_{V_1} + \|\vec{u}\|_{V_1}) \|\vec{h}\|_{V_1} .$$

Relation (3.47) follows from (3.49) bearing in mind the assumptions made on the coefficients and (3.42).

The function $\vec{u}(t)$ therefore satisfies, by (3.19), (3.44), (3.45), (3.46), (3.47) equation (3.15), $\forall \vec{h}(t)$ given by (3.43). As the space of these functions is dense in that of the "test functions" of (3.15), $\vec{u}(t)$ is a solution on $0 \,|\!\!-\!\!|\, \bar{t}$ of system (1.1) satisfying the given boundary conditions. Moreover, by (3.37), the functions $\vec{u}_n(t)$ are $V_{\sigma-1}$–equally continuous on $0 \,|\!\!-\!\!|\, \bar{t}$; as the embedding of $V_{\sigma-1}$ in V_ϱ, with $\rho < \sigma - 1$, is compact, the functions $\vec{u}_n(t)$ have, on the other hand, V_ϱ–relatively compact range $\forall t \in 0 \,|\!\!-\!\!|\, \bar{t}$.

Hence, by the (vectorial) theorem of Ascoli–Arzelà,

$$\lim_{n \to \infty} \vec{u}_n(t) \underset{V_\varrho}{=} \vec{u}(t) \tag{3.50}$$

uniformly on $0 \,|\!\!-\!\!|\, \bar{t}$.

By (3.17), (3.20), (3.50) we obtain therefore

$$\vec{u}(0) \underset{V_\varrho}{=} \vec{z} .$$

Bearing in mind relations (3.31), we obtain also,

$$\vec{u}(t) \in L^\infty(0 \,|\!\!-\!\!|\, \bar{t} \,; V_\sigma) \cap L^2(0 \,|\!\!-\!\!|\, \bar{t} \,; V_{\sigma+1}) .$$

The existence theorem is then completely proved.

As we shall prove in the subsequent note III, the solution found is also unique.