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Demetrio Mangeron, Mehmet Namik Oğuztöreli

## On a class of integro-differential equations. Nota II

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**Analisi matematica.** — *On a class of integro-differential equations.* Nota II di DEMETRIO MANGERON (*) (**) e MEHMET NAMIK OĞUZTÖRELI (***) (****), presentata (*****) dal Socio M. PICONE.

RIASSUNTO. — In questo lavoro gli AA. continuano il loro studio concernente un problema di valori al contorno spettante ad un'equazione integro-differenziale a derivate parziali del second'ordine considerata in [1]. Si determina la soluzione per certi valori dei parametri.

In this Note we continue our investigation on a boundary value problem for an integro-partial differential equation of the second order considered in [1], and we establish the solution for certain values of the parameters.

## 1. INTRODUCTION.

In [1] we have considered the integro-partial differential equation

$$\lambda \frac{\partial^2 u(x,y)}{\partial x^2} + (\lambda - 1) u(x,y) = g_0(x,y) + \mu \iint_R K(x,y;\xi,\eta) u(\xi,\eta) \, d\xi \, d\eta \tag{1.1}$$

subject to the boundary conditions

$$u(0,y) = u(1,y) = 0 \qquad \text{for} \quad 0 \leq y \leq 1 , \tag{1.2}$$

where $R = \{(x,y) \mid 0 \leq x , y \leq 1\}$, $\lambda$ and $\mu$ are real parameters, $g_0(x,y)$ and $K(x,y;\xi,\eta)$ are given functions which are continuous on R and $R \times R$ respectively, and $u(x,y)$ is the unknown function. It was shown in [1] that the boundary value problem (1.1)-(1.2) has always a unique solution, for sufficiently small $\mu$ and for any $\lambda \neq 0$, $\lambda \neq (1 - k^2 \pi^2)^{-1}$, in the form of a power series in $\mu$ whose coefficients satisfy a recurrent system of integro-partial differential equations with homogeneous boundary conditions.

In this paper we establish the solution of the above boundary value problem in the form of a power series in $(\lambda - 1)$ and $\lambda^{-1}$ for sufficiently small $\mu$.

(*) Polytechnic Institute of Jassy, Jassy-Romania. At present Visiting Professor in the Department of Mathematics, University of Alberta, Edmonton 7, Alberta, Canada.

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(***) Department of Mathematics, University of Alberta, Edmonton 7, Alberta, Canada.

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## 2. Expansion in Powers of $(\lambda - 1)$.

In this section we seek a solution to the boundary value problem (1.1)–(1.2) in the form

$$u(x, y) = \sum_{n=0} (\lambda - 1)^n u_n(x, y), \tag{2.1}$$

where

$$u_n(0, y) = u_n(1, y) = 0 \qquad \text{for} \quad 0 \leq y \leq 1, \qquad n = 0, 1, 2, \cdots \tag{2.2}$$

for sufficiently small $\mu$. To do so, we substitute Eq. (2.1) in Eq. (1.1). Comparing the coefficients of $(\lambda - 1)^n$ on both sides, we find

$$\frac{\partial^2 u_n(x,y)}{\partial x^2} = g_n(x, y) + \mu \iint_R K(x, y; \xi, \eta)\, u_n(\xi, \eta)\, d\xi\, d\eta, \tag{2.3}$$

where

$$g_n(x, y) = - u_{n-1}(x, y) - \frac{\partial^2 u_{n-1}(x,y)}{\partial x^2}, \qquad n = 1, 2, 3, \cdots. \tag{2.4}$$

We can easily show that solving Eq. (2.3) subject to the boundary conditions (2.2) is equivalent to solving the following Fredholm integral equation:

$$u_n(x,y) = P_n(x,y) + \mu \iint_R H(x, y; \xi, \eta)\, u_n(\xi, \eta)\, d\xi\, d\eta, \quad n = 0, 1, 2, \cdots \tag{2.5}$$

where

$$P_n(x, y) = - \int_0^1 G(x, \sigma)\, g_n(\sigma, y)\, d\sigma, \qquad n = 0, 1, 2, \cdots \tag{2.6}$$

and $G(x, \sigma)$ and $H(x, y; \xi, \eta)$ are as in [1]. Eq. (2.5) has a unique solution for sufficiently small $\mu$, and this solution is given by the formula

$$u_n(x,y) = P_n(x,y) + \mu \iint_R M(x, y; \xi, \eta; \mu)\, P_n(\xi, \eta)\, d\xi\, d\eta, \quad n = 0, 1, 2, \cdots \tag{2.7}$$

where $M(x, y; \xi, \eta; \mu)$ is the resolvent of the kernel $H(x, y; \xi, \eta)$.

Put

$$M_\mu = \left\| \iint_R |M(x, y; \xi, \eta; \mu)|\, d\xi\, d\eta \right\|, \tag{2.8}$$

the notation $\|\cdot\|$ being as in [1]. Then, according to Eq. (2.7), we may write

$$\| u_n \| \leq (1 + |\mu| M_\mu) \| P_n \|, \qquad n = 0, 1, 2, \cdots. \tag{2.9}$$

Further, by virtue of Eq. (2.6), we have

$$\| P_n \| \leq \frac{1}{8} \| g_n \|, \qquad n = 0, 1, 2, \cdots. \tag{2.10}$$

On the other hand, from Eq. (2.3), we find

$$\left\| \frac{\partial^2 u_n}{\partial x^2} \right\| \leq \| g_n \| + | \mu | K \| u_n \| , \qquad n = 0, 1, 2, \cdots \tag{2.11}$$

K being as in [1]. Finally, according to Eqs. (2.5) and (2.11), we have

$$\| g_n \| \leq \| g_{n-1} \| + (1 + | \mu | K) \| u_{n-1} \| , \qquad n = 1, 2, 3, \cdots . \tag{2.12}$$

Consequently

$$\| g_n \| \leq \left[ 1 + \frac{(1 + | \mu | K)(1 + | \mu | M_\mu)}{8} \right] \| g_{n-1} \| , \qquad n = 1, 2, 3, \cdots . \tag{2.13}$$

Clearly, the numerical series

$$\sum_{n=0}^{\infty} \frac{1 + | \mu | M_\mu}{8} \| g_n \| \, | \lambda - 1 |^n \tag{2.14}$$

dominates the series (2.1). The series (2.14) converges for

$$| \lambda - 1 | < \frac{8}{(1 + | \mu | M_\mu)(1 + | \mu | K) + 8} \tag{2.15}$$

by virtue of inequalities (2.13). Hence the series (2.1) converges absolutely and uniformly on the square R for sufficiently small $\mu$ and for $\lambda$ satisfying the inequality (2.15). A similar analysis can be carried out for the series $\sum_{n=0}^{\infty} (\lambda - 1)^n \frac{\partial^2 u_n(x, y)}{\partial x^2}$. Thus, the series (2.1) is a solution of the boundary value problem (1.1)–(1.2). The uniqueness of this solution is assured by the results of [1].

## 3. SOLUTION FOR LARGE $\lambda$.

We now construct the solution of the boundary value problem (1.1)–(1.2) for large $\lambda$. For this purpose we seek a solution of the form

$$\begin{cases} u(x, y) = \sum_{n=1}^{\infty} \lambda^{-n} u_n(x, y) , \\ u_n(0, y) = u_n(1, y) = 0 \quad \text{for} \quad 0 \leq y \leq 1 , \qquad n = 1, 2, 3, \cdots . \end{cases} \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (1.1), and comparing the coefficients of $\lambda^{-n}$ on both sides, we find

$$\begin{cases} \dfrac{\partial^2 u_1(x, y)}{\partial x^2} + u_1(x, y) = g_0(x, y) , \\ \dfrac{\partial^2 u_n(x, y)}{\partial x^2} + u_n(x, y) = u_{n-1}(x, y) + \mu \iint\limits_R K(x, y; \xi, \eta) u_{n-1}(\xi, \eta) \, d\xi \, d\eta , \\ \qquad n = 2, 3, \cdots . \end{cases} \tag{3.2}$$

Solving Eqs. (3.2) subject to the homogeneous boundary conditions (3.1) is equivalent to the solving the integral equations

$$(3.3) \qquad u_n(x,y) = Q_n(x,y) + \int_0^1 G(x,\sigma)\, u_n(\sigma,y)\, d\sigma, \qquad n = 1, 2, 3, \cdots$$

where

$$(3.4) \quad \begin{cases} Q_1(x,y) = -\displaystyle\int_0^1 G(x,\sigma)\, g_0(\sigma,y)\, d\sigma, \\ Q_n(x,y) = -\displaystyle\int_0^1 G(x,\sigma)\, [u_{n-1}(\sigma,y) + \\ \qquad + \mu \displaystyle\iint_R K(\sigma, y; \xi, \eta)\, u_{n-1}(\xi,\eta)\, d\xi\, d\eta]\, d\sigma, \qquad n = 2, 3, \cdots. \end{cases}$$

Note that 1 is not an eigenvalue of the kernel $G(x,\sigma)$. Accordingly, Eq. (3.3) has a unique solution. This solution is given by the formula

$$(3.5) \qquad u_n(x,y) = Q_n(x,y) + \int_0^1 \Gamma(x,\sigma;1)\, Q_n(\sigma,y)\, d\sigma, \qquad n = 1, 2, 3, \cdots$$

where $\Gamma(x,\sigma;\nu)$ is the resolvent of the kernel $G(x,\sigma)$ (cfr. [1]).

We can easily show that

$$(3.6) \qquad \begin{cases} \| u_1 \| \le \dfrac{1+\Gamma_1}{8} \| g_0 \|, \\ \left\| \dfrac{\partial^2 u_1}{\partial x^2} \right\| \le \dfrac{9+\Gamma_1}{8} \| g_0 \|, \end{cases}$$

and

$$(3.7) \qquad \begin{cases} \| u_n \| \le \dfrac{(1+\Gamma_1)^n (1+|\mu| K)^{n-1}}{8^n} \| g_0 \|, \\ \left\| \dfrac{\partial^2 u_n}{\partial x^2} \right\| \le \dfrac{(9+\Gamma_1)(1+\Gamma_1)^{n-1}(1+|\mu| K)^n}{8^{n-1}} \| g_0 \| \end{cases} \qquad n = 2, 3, \cdots.$$

Clearly, the series

$$(3.8) \qquad \sum_{n=1}^{\infty} \frac{(1+\Gamma_1)^n (1+|\mu| K)^{n-1}}{8^n} \| g_0 \| \, |\lambda|^{-n},$$

$$\sum_{n=1}^{\infty} \frac{(9+\Gamma_1)(1+\Gamma_1)^{n-1}(1+|\mu| K)^n}{8^{n-1}} \| g_0 \| \, |\lambda|^{-n}$$

dominate the series (3.1) and $\sum_{n=1}^{\infty} \lambda^{-n} \dfrac{\partial^2 u_n(x,y)}{\partial x^2}$, respectively, and both converge for

$$(3.9) \qquad |\lambda| > \frac{(1+\Gamma_1)(1+|\mu| K)}{8}.$$

Hence $\sum_{n=1}^{\infty} \lambda^{-n} u_n(x, y)$ and $\sum_{n=1}^{\infty} \lambda^{-n} \frac{\partial^2 u_n(x, y)}{\partial x^2}$ are convergent absolutely and uniformly on the square R for sufficiently small $\mu$ and for $\lambda$ satisfying the condition (3.9). The uniform and absolute convergence of the series $\sum_{n=1}^{\infty} \lambda^{-n} \frac{\partial u_n(x, y)}{\partial x}$ can be established similarly. Thus, the series (3.1) is the unique solution of the boundary value problem (1.1)–(1.2) for large $\lambda$.

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