Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Kurt Kreith

Sturm Theory for Nonlinear Elliptic Equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **48** (1970), n.2, p. 140–143.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1970_8_48_2_140_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Matematica. — *Sturm Theory for Nonlinear Elliptic Equations.* Nota di KURT KREITH, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Viene impiegata la cosidetta « Picone idendity » per stabilire nuovi teoremi d'oscillazione concernenti le soluzioni di equazioni a derivate parziali del secondo ordine dotate di una certa *non* linearità.

Sturm's comparison theorem deals with functions $u(x)$ and $v(x)$ which are non-trivial solutions of

$$u'' + p(x)\,u = 0\,, \tag{1}$$

$$v'' + q(x)\,v = 0\,, \tag{2}$$

respectively. Given that $p(x) \leq q(x)$, Sturm's theorem asserts that " $v(x)$ oscillates faster than $u(x)$ " in the sense that if x_1 and x_2 are zeros of $u(x)$, then $v(x)$ has a zero in $[x_1, x_2]$.

In order to motivate a generalization to nonlinear equations, suppose that $p(x) > 0$ so that u'' has the opposite sign to u. In this situation, the larger $p(x)$ the faster $u(x)$ is driven back to the line $u = 0$ due to the effect of the second derivative: $u'' = -p(x)\,u$.

This point of view suggests that there are ways of increasing the rate of oscillation of solutions of second order equations by including nonlinear terms. For example, if $q(x) \geq p(x)$ and $r(x)$ is positive, then it is reasonable to expect that solutions of $v'' + q(x)\,v + r(x)\,v^3 = 0$ will oscillate faster than solutions of (1). More generally, if $f(x, v)$ is any sufficiently regular function satisfying $f(x, v) \geq 0$ for $v \geq 0$ and $f(x, v) \leq 0$ for $v \leq 0$, then one would expect that solutions of $v'' + q(x)\,v + f(x, v) = 0$ will oscillate faster than solutions of (1), while solution of $v'' + p(x)\,v - f(x, v) = 0$ will oscillate more slowly.

A comparison theorem along these lines was proven by Taam [1] for certain nonlinear second order ordinary differential equations. We shall prove a comparison theorem for nonlinear elliptic equations in R^n which includes Taam's comparison theorem as a special case when $n = 1$.

Our principal result deals with non-trivial solutions of sufficiently regular nonlinear elliptic boundary value problems of the form

$$\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\,\frac{\partial u}{\partial x_i}\right) + c(x, u) = 0 \qquad \text{in } G \\ &\frac{\partial u}{\partial \nu} + s(x)\,u = 0 \qquad \text{on } \partial G \end{aligned} \tag{3}$$

(*) Nella seduta del 10 gennaio 1970.

and

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) + \gamma(x, v) = 0 \qquad \text{in} \quad G$$

(4)

$$\frac{\partial v}{\partial \nu} + \sigma(x)\, v = \qquad \text{on} \quad \partial G\,.$$

Here $x = (x_1, \cdots, x_n)$ and

$$\frac{\partial u}{\partial \nu} = \Sigma a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \nu}{\partial x_j}\,,$$

$$\frac{\partial v}{\partial \nu} = \Sigma \alpha_{ij}\frac{\partial v}{\partial x_i}\frac{\partial \nu}{\partial x_j}\,,$$

where $\frac{\partial \nu}{\partial x_j}$ denotes the cosine of the angle between the exterior normal ν and the positive x_j–axis. We follow the convention of allowing $s(x) = +\infty$ to denote the boundary condition $u(x) = 0$. It is assumed that ∂G has a piecewise smooth normal and that the a_{ij} and α_{ij} are of class C' in $\bar{G}$. The functions $c(x, u)$ and $\gamma(x, v)$ are to be continuous in $G \times (-\infty, \infty)$ and satisfy

$$c(x, 0) = \gamma(x, 0) = 0\,.$$

Furthermore the following four limits

$$\lim_{u \downarrow 0} \frac{c(x, u)}{u}\,, \qquad \lim_{u \uparrow 0} \frac{c(x, u)}{u}\,, \qquad \lim_{v \downarrow 0} \frac{\gamma(x, v)}{v}\,, \qquad \lim_{v \uparrow 0} \frac{\gamma(x, v)}{v}$$

are assumed to exist for every $x \in \bar{G}$.

THEOREM I. *Suppose* $u(x)$ *and* $v(x)$ *are solutions of* (3) *and* (4), *respectively. If*

(i) $\Sigma a_{ij}\xi_i\xi_j \geq \Sigma \alpha_{ij}\xi_i\xi_j > 0$ for every $x \in G$ and all real n–tuples $(\xi_1, \cdots, \xi_n)$;

(ii) $\frac{\gamma(x, v)}{v} \geq \frac{c(x, u)}{u}$ for every $x \in G$ and all u, v in $(-\infty, \infty)$;

(iii) $s(x) \geq \sigma(x)$ on ∂G,

then $v(x)$ *has a zero in* $\bar{G}$ *or else* $v(x)$ *is a scalar multiple of* $u(x)$.

Proof. The proof will depend on the following generalized Picone identity [2], [3]: If $v(x) \neq 0$ in $\bar{G}$, then

(5)
$$\sum_j \frac{\partial}{\partial x_j}\left[\frac{u}{v}\left(v\sum_i a_{ij}\frac{\partial u}{\partial x_i} - u\sum_i \alpha_{ij}\frac{\partial v}{\partial x_i}\right)\right] = u\sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)$$
$$-\frac{u^2}{v}\sum_{i,j}\frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) + \sum_{i,j}(a_{ij} - \alpha_{ij})\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}$$
$$+\sum_{i,j}\alpha_{ij}\left(\frac{\partial u}{\partial x_i} - \frac{u}{v}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j} - \frac{u}{v}\frac{\partial v}{\partial x_j}\right),$$

where the last two terms in (5) are non-negative by (i). Using the differential equations in (3) and (4) to simplify the first two terms on the right side of (5), integrating over G, and applying Green's theorem yields

$$\int\limits_{\partial G} \left[\frac{u}{v} \left(v \sum_i a_{ij} \frac{\partial u}{\partial x_i} - u \sum_i \alpha_{ij} \frac{\partial v}{\partial x_i} \right) \right] \frac{\partial \nu}{\partial x_i} \, ds \geq \int\limits_{\bar{G}} u^2 \left[\frac{\gamma(x, v)}{v} - \frac{c(x, u)}{u} \right] dx$$

with equality if and only if $v(x)$ is a scalar multiple of $u(x)$. Using the boundary condition in (3) and (4) we get [1]

$$(6) \qquad - \int\limits_{\partial G} [s(x) - \sigma(x)] \, u^2 \, ds \geq u^2 \left[\frac{\gamma(x, v)}{v} - \frac{c(x, u)}{u} \right].$$

However by (ii) the right side of (6) is non-negative while by (iii) the left side is non-positive. Therefore the assumption $v(x) \neq 0$ in $\bar{G}$ is tenable only if $v(x)$ is a scalar multiple of $u(x)$.

COROLLARY. *Suppose* $u(x)$ *and* $v(x)$ *are as in Theorem* 1 *and that conditions* (i) *and* (ii) *are satisfied. If* G *is a nodal domain for* $u(x)$, *then* $v(x)$ *has a zero in* $\bar{G}$.

Proof. This is the situation in which "$s(x) \equiv +\infty$ on ∂G" so that (iii) is automatically satisfied. More precisely, since $u(x) \equiv 0$ on ∂G, the boundary integral in (6) vanishes and the conclusion of Theorem 1 remains valid.

The following are examples of pairs of functions $c(x, u)$ and $\gamma(x, v)$ which satisfy condition (ii) of Theorem 1.

1) $c(x, u) = u - u^5,$

$\gamma(x, v) = 2v + v^3.$

2) $c(x, u) = \sum_{k=1}^{M} p_k(x) \, u^{2k-1},$

$\gamma(x, v) = \sum_{k=1}^{N} q_k(x) \, v^{2k-1},$

where $q_1(x) \geq p_1(x)$, $p_k(x) \leq 0$ for $k \geq 2$ and $q_k(x) \geq 0$ for $k \geq 2$.

3) $c(x, u) = -\sinh u,$

$\gamma(x, v) = -v \quad$ for $\quad v \leq 0,$

$\qquad = v^3 \quad$ for $\quad v > 0.$

Oscillation theorems for nonlinear singular elliptic equations of the form

$$(7) \qquad \Sigma \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial v}{\partial x_i} \right) + \gamma(x, v) = 0 \, ; \qquad x \in G$$

(1) In case $s(x) = \infty$ on part of ∂G, we are justified in omitting that portion of ∂G in the boundary integral below.

follow readily from the Corollary by comparing (7) with the singular linear equation

$$(8) \qquad \frac{\partial}{\partial x_n}\left(a(x_n)\frac{\partial u}{\partial x_n}\right)+\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+c(x_n)\,u=0\,; \qquad x \in G$$

under the assumption that some solution of (8) has a sequence of nodal domains accumulating at a singular surface $S \subset \partial G$. By assuming further that hypotheses (i) and (ii) of Theorem 1 are satisfied near S, one can conclude that (7) is " oscillatory at S ". This procedure is exactly analogous to that used by the author in [4] to derive oscillation theorems for linear elliptic equations and will not be repeated here. Rather we shall simply state the results (using the terminology introduced in [4]). It is assumed that $G \subset \{x \mid x_n > 0\}$ and that the singular part of ∂G satisfies $S \subset \{x \mid x_n = 0\}$.

THEOREM 2. *Suppose $u(x)$ and $v(x)$ are solutions of* (7) *and* (8), *respectively, and that*

(i) $\Sigma a_{ij}\,\xi_i\,\xi_j \geq \Sigma \alpha_{ij}\,\xi_i\,\xi_j > 0$ for all x near S,

(ii) $\dfrac{\gamma(x,v)}{v} \geq c(x_n)$ for all x near S and $-\infty < v < \infty$.

If for some $\varepsilon > 0$ the equation

$$\frac{d}{dt}\left(a(t)\frac{dw}{dt}\right)+\left[c(t)-(\mu_0+\varepsilon)\right]w$$

is oscillatory at $t = 0$, then every solution of (7) *is weakly oscillatory at* S (*in the sense that if* H *is an open set containing* S, *then $u(x)$ has a zero in* $H \cap G$).

THEOREM 3. *Under the hypotheses of Theorem* 2, *if*

$$\frac{d}{dt}\left(a(t)\frac{dw}{dt}\right)+\left[c(t)+M\right]w=0$$

is oscillatory at $t = 0$ for every real number M, *then every solution of* (7) *is strongly oscillatory at* S (*in the sense that if $x_0 \in S$ and* H *is a neighborhood of x_0, then $u(x)$ has a zero in* $H \cap G$).

REFERENCES.

[1] C. T. TAAM, *An extension of Osgood's oscillation theorem for a nonlinear differential equation*, « Proc. Amer. Math. Soc. », *5*, 705–715 (1954).

[2] M. PICONE, *Un teorema sulle soluzioni delle equazioni lineari ellittiche autoaggiunte alle derivate parziali del secondo–ordine*, « Atti Accad. Lincei », *20*, 213–219 (1911).

[3] K. KREITH, *A generalized Picone identity*, « Atti Accad. Lincei », *45*, 217–220 (1968).

[4] K. KREITH, *Oscillation theorems for elliptic equations*, « Proc. Amer. Math. Soc. », *15*, 341–344 (1964).