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A Picone identity for non—self—adjoint elliptic operators

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Matematica. — *A Picone identity for non-self-adjoint elliptic operators* (*). Nota di D. R. DUNNINGER, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota è ottenuta una estensione della identità di Picone da questi data per gli operatori ellittici autoaggiunti al caso in cui gli operatori non sono tali. Impiegando questa estensione vi si ottengono teoremi tipo Sturm per operatori ellittici non autoaggiunti con condizioni lineari e omogenee al contorno.

1. INTRODUCTION.

Comparison theorems of Sturm's type will be considered for linear non-self-adjoint elliptic inequalities under linear homogeneous boundary conditions. Results of this nature for homogeneous Dirichlet boundary conditions have been considered by Swanson [1] who based his results on a variational-type lemma. The results considered here will depend upon a new Picone-type identity for non-self-adjoint elliptic operators which generalizes the Picone identity [2] for self-adjoint elliptic operators. In particular, Swanson's variational lemma will be seen to be an easy consequence of the identity.

2. NOTATION.

Let R be a bounded domain in n-dimensional Euclidean space E^n with a piecewise smooth boundary ∂R. Points in E^n are denoted by $x = (x^1, x^2, \cdots, x^n)$ and differentiation with respect to x^i is denoted by D_i, $(i = 1, 2, \cdots, n)$.

Consider the linear differential operators l, L defined by

$$lu \equiv \sum_{i,j=1}^{n} D_i (a_{ij} D_j u) + 2 \sum_{i=1}^{n} b_i D_i u + cu , \tag{2.1}$$

$$Lv \equiv \sum_{i,j=1}^{n} D_i (A_{ij} D_j v) + 2 \sum_{i=1}^{n} B_i D_i v + Cv , \tag{2.2}$$

respectively, where the domains $\mathfrak{D}_l$, $\mathfrak{D}_L$ of l, L, respectively, are defined to be the sets of all complex-valued functions $u \in C'(R \cup \partial R)$, such that all derivatives appearing in lu and Lv exist and are continuous at every point in R. The coefficients in (2.1) and (2.2) are assumed to be

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real and continuous in $R \cup \partial R$. The matrices (a_{ij}) and (A_{ij}) are assumed symmetric and positive definite (ellipticity condition) in R.

Definition 2.1. (Swanson [1]). An L–subsolution (–supersolution) is a real valued function $v \in \mathfrak{D}_L$ which satisfies $Lv \leq 0$ ($Lv \geq 0$) at every point in R.

3. The Picone identity for non-self-adjoint elliptic operators.

Let $u(x) \in \mathfrak{D}_l$ and let $v(x) \in \mathfrak{D}_L$ be real-valued. If $v(x) \neq 0$ in $R \cup \partial R$, then

$$\begin{aligned}(3.1)\qquad & Re \sum_i D_i \left[\rho\left(v \sum_j a_{ij} D_j \bar{u} - \bar{u} \sum_j A_{ij} D_j v\right)\right] \\ & = \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2\, Re \sum_i u (b_i - B_i) D_i \bar{u} + (C - c - G) |u|^2 \\ & + \sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) - 2\, Re \sum_i u B_i (v D_i \bar{\rho}) + G\, |u|^2 \\ & + Re\, [\rho\, (v l \bar{u} - \bar{u} L v)]\,,\end{aligned}$$

where a bar denotes complex conjugation, $\rho = u/v$ and the continuous function G is chosen so that the hermitian form

$$Q[X] = \sum_{i,j} A_{ij} X^i \overline{X^j} - 2 \sum_i B_i\, Re\, (\overline{X^i}\, X^{n+1}) + G\, |X^{n+1}|^2,$$

where

$$X^i = v D_i \rho \quad , \quad X^{n+1} = u\,,$$

is positive semidefinite. It can be shown [3], that a necessary and sufficient condition for Q [X] to be positive semidefinite is

$$(3.2)\qquad G \det (A_{ij}) \geq \sum_i B_i B_i^*\,,$$

where B_i^* denotes the cofactor of $-B_i$ in the matrix Q associated with Q [X]:

$$Q = \begin{pmatrix} (A_{ij}) & -(B_i) \\ -(B_i)^T & G \end{pmatrix}.$$

The proof of the identity is indicated below, and is almost self-explanatory. Let m, M denote the divergence part of the operators l, L, respectively. Then

$$\begin{aligned}(3.3)\qquad & \sum_i D_i \left[\rho\left(v \sum_j a_{ij} D_j \bar{u} - \bar{u} \sum_j A_{ij} D_j v\right)\right] \\ & = \sum_{i,j} a_{ij} D_i u D_j \bar{u} + \frac{|u|^2}{v} \sum_{i,j} A_{ij} D_i v D_j v - \frac{1}{v} \sum_{i,j} A_{ij} D_j v D_i |u|^2 + \\ & \qquad + \rho\, (v m \bar{u} - \bar{u} M v) \\ & = \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} + \sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) + \rho\, (v m \bar{u} - \bar{u} M v)\end{aligned}$$

$$= \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2u \sum_i b_i D_i \bar{u} + (C - c) |u|^2$$

$$+ \sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) + 2 \frac{|u|^2}{v} \sum_i B_i D_i v + \rho (v l \bar{u} - \bar{u} L v)$$

$$= \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2u \sum_i (b_i - B_i) D_i \bar{u} + (C - c - G) |u|^2$$

$$+ \sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) - 2u \sum_i B_i (v D_i \bar{\rho}) + G |u|^2$$

$$+ \rho (v l \bar{u} - \bar{u} L v) .$$

The identity (3.1) is realized by taking the real part of (3.3).

Note that in the self-adjoint case $b_i = B_i = 0$ $(i = 1, 2, \cdots, n)$ and $G = 0$, (3.1) generalizes the Picone identity slightly in that we have allowed u to be complex-valued.

In what follows we shall need (3.1) in the following integral form obtained by integrating (3.1) over R and applying Green's theorem:

$$\text{(3.4)} \qquad \operatorname{Re} \int_{\partial R} \rho \left(v \frac{\partial \bar{u}}{\partial \sigma} - \bar{u} \frac{\partial v}{\partial \nu} \right) ds$$

$$= \int_R \Big[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2 \operatorname{Re} \Big(u \sum_i (b_i - B_i) D_i \bar{u} \Big) + $$
$$+ (C - c - G) |u|^2 \Big] dx$$

$$+ \int_R \Big[\sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) - 2 \operatorname{Re} \Big(u \sum_i B_i v D_i \bar{\rho} \Big) + G |u|^2 \Big] dx$$

$$+ \operatorname{Re} \int_R \rho (v l \bar{u} - \bar{u} L v) \, dx ,$$

where

$$\frac{\partial u}{\partial \sigma} = \sum_{i,j} n_j a_{ij} D_i u \quad , \quad \frac{\partial v}{\partial \nu} = \sum_{i,j} n_j A_{ij} D_i v ,$$

are the conormal derivatives associated with the operators l, L, respectively, (n_j) being the unit exterior normal vector to the boundary ∂R.

For later reference, we note that the condition (3.2) implies that the second integral on the right side of (3.4) is nonnegative.

4. Comparison theorems.

THEOREM 4.1. *Given* $a(x)$ *and* $A(x)$ *are real-valued functions which are continuous on* ∂R. *Suppose* G *satisfies* (3.2) *in* R. *If there exists a nontrivial* $u \in \mathfrak{D}_l$ *which satisfies*

(i) $\operatorname{Re} (u\, l \bar{u}) \geq 0$ in R,

(ii) $u = 0$ on Γ_1 , $\frac{\partial u}{\partial \sigma} + a(x)\, u = 0$ on Γ_2 , $\partial R = \Gamma_1 \cup \Gamma_2$,

$$\text{(4.1)} \quad V[u] = \int_R \Big[\sum_{i,j} (a_{ij} - A_{ij})\, D_i u D_j \bar{u} - 2\,\mathrm{Re} \sum_i u\,(b_i - B_i)\, D_i \bar{u} + (C - c - G)\,|u|^2\Big]\, dx + \int_{\Gamma_2} (a - A)\,|u|^2\, ds > 0,$$

then every L*–subsolution (–supersolution)* v *which is positive (negative) at some point in* R *and satisfies*

(iii) $\frac{\partial v}{\partial \nu} + A(x) \cdot v = 0$ on Γ_2,

must vanish at some point in $R \cup \partial R$.

In particular, every real solution of $Lv = 0$ *which statisfies* (iii) *must vanish at some point in* $R \cup \partial R$. *In the self-adjoint case* $b_i = B_i = 0$ $(i = 1, 2, \cdots, n)$ *and* $G = 0$, *the same conclusions are valid when the hypothesis* $V[u] > 0$ *is relaxed to* $V[u] \geq 0$, *provided* Γ_1 *is not empty.*

Proof. Assume on the contrary that there exists an L–subsolution (–supersolution) $v \neq 0$ in $R \cup \partial R$. Consequently, $v > 0$ $(v < 0)$ in $R \cup \partial R$ and the integral identity (3.4) therefore holds and becomes in view of the above hypotheses

$$\text{(4.2)} \quad 0 = \mathrm{Re} \int_{\Gamma_1} \rho \left(v \frac{\partial \bar{u}}{\partial \sigma} - \bar{u} \frac{\partial v}{\partial \nu}\right) ds \geq V[u] + \mathrm{Re} \int_R \left(u l \bar{u} - \frac{|u|^2}{v} L v\right) dx \geq V[u],$$

which contradicts (4.1).

To prove the second statement we note that a solution of $Lv = 0$ which satisfies (iii) and never vanishes in $R \cup \partial R$ will be either an L–subsolution or an L–supersolution which is positive or negative, respectively, at some point in R.

To prove the third statement it suffices to show that the second integral on the right hand side of (3.4) is positive, since the conclusions then follow from the obvious modification of the inequalities in (4.2). In view of the positive definiteness of (A_{ij}) it follows that

$$\text{(4.3)} \quad \int_R \sum_{i,j} A_{ij}\,(v D_i \rho)\,(v D_j \bar{\rho})\, dx \geq 0,$$

with equality holding if and only if $D_i \rho = 0$ for each $i = 1, 2, \cdots, n$. This implies that u and v are linearly dependent, a condition which cannot hold since $u = 0$ on Γ_1, whereas $v \neq 0$ on Γ_1. Therefore, the integral in (4.3) is positive.

Remarks.

1. If Γ_1 is empty, then the conclusions with respect to the self-adjoint case have to be modified and one can only conclude that either v vanishes at a point in $R \cup \partial R$ or u and v are linearly dependent.

2. If either of the inequalities (i), (3.2) is replaced by a strict inequality then the conclusions hold when the hypothesis $V[u] > 0$ is weakened to $V[u] \geq 0$. In particular, in the self-adjoint case Γ_1 is allowed to be empty provided a strict inequality holds in (i).

We note that strict inequality in (3.2) implies that $Q[X]$ is positive definite and is equal to zero if and only if $u \equiv 0$ in $R \cup \partial R$.

3. Suppose Γ_1 is empty, $C \geq 0$ and ∂R has the property that at each boundary point $\hat{x}$, there is a hypersphere lying entirely in $R \cup \partial R$ which has $\hat{x}$ as a boundary point. It can then be concluded that v must vanish in R. Indeed, assume v vanishes at a point $\hat{x} \in \partial R$, but that v does not vanish in R. The result of Hopf [4] implies $\partial v/\partial \nu \neq 0$ at $\hat{x}$ contradicting the boundary condition (iii).

4. Our results can also be formulated in terms of the nonlinear equations

$$lu = f(x, u)\, u\,, \tag{4.4}$$

$$Lv = F(x, v)\, v\,, \tag{4.5}$$

where f and F are given continuous functions (F being real-valued) in $R \cup \partial R$. The hypothesis that $\mathrm{Re}\,(f(x, \bar{u})) \geq F(x, v)$ assures that Theorem 4.1 applies.

As an application of Theorem 4.1 (in particular Remark 4) we obtain a monotonicity principle for the smallest eigenvalues of the eigenvalue problems

$$\begin{aligned} & lu + \lambda u = 0 && \text{in } R', \\ & u = 0 && \text{on } \partial R', \end{aligned} \tag{4.6}$$

$$\begin{aligned} & Lv + \mu v = 0 && \text{in } R, \\ & B(v) = 0 && \text{on } \partial R. \end{aligned} \tag{4.7}$$

Here, it is assumed that R' is a domain in E^n such that $R' \cup \partial R' \subset R$, the operators l, L are self-adjoint, and $B(v)$ is any linear boundary condition which insures that the eigenfunction corresponding to the smallest eigenvalue of (4.7) does not vanish in R (see [5], p. 452).

THEOREM 4.2. *Let λ and μ be the smallest eigenvalues of* (4.6) *and* (4.7), *respectively, with corresponding eigenfunctions u and v. If u satisfies* $V[u] \geq 0$, *then $\lambda > \mu$. (Here, the integral in* $V[u]$ *is over* R').

Proof. Since v does not vanish in R, the integral identity (3.4) is valid for the domain R'. If we assume on the contrary that $\lambda \leq \mu$, then Remark 4 with $f = -\lambda$, $F = -\mu$ implies that v must vanish in $R' \cup \partial R'$ which is a contradiction, hence $\lambda > \mu$.

5. A VARIATIONAL-TYPE THEOREM.

THEOREM 5.1. *Given* $A(x)$ *is a real-valued function which is continuous on* ∂R. *Suppose* G *satisfies* (3.2) *in* R. *If there exists a nontrivial complex-valued function* $u \in C'(R \cup \partial R)$, $u = 0$ *on* Γ_1 *such that*

$$M[u] = \int_R \Big[\sum_{i,j} A_{ij} D_i u D_j \bar{u} - 2 \operatorname{Re} \sum_i u B_i D_i \bar{u} + (G - C)|u|^2\Big] dx + \int_{\Gamma_2} A|u|^2 ds < 0,$$

where $\partial R = \Gamma_1 \cup \Gamma_2$, *then every* L*–subsolution (–supersolution)* v *which is positive (negative) at some point in* R *and satisfies*

$$\text{(iii)} \quad \frac{\partial v}{\partial \nu} + A(x)\, v = 0 \quad \text{on } \Gamma_2,$$

must vanish at some point in $R \cup \partial R$.

In particular, every real solution of $Lv = 0$ *which statisfies* (iii) *must vanish at some point in* $R \cup \partial R$. *In the self-adjoint case* $b_i = B_i = 0$ $(i = 1, 2, \cdots, n)$ *and* $G = 0$, *the same conclusions are valid when the hypothesis* $M[u] < 0$ *is relaxed to* $M[u] \leq 0$, *provided* Γ_1 *is not empty.*

Proof. Setting $a_{ij} = b_i = c = 0$ $(i, j = 1, 2, \cdots, n)$, in (3.4) we obtain the integral identity

$$\int_{\partial R} \frac{|u|^2}{v} \frac{\partial v}{\partial \nu} ds = \int_R \Big[\sum_{i,j} A_{ij} D_i u D_j \bar{u} - 2 \operatorname{Re}\Big(u \sum_i B_i D_i \bar{u}\Big) + (G - C)|u|^2\Big] dx$$

$$- \int_R \Big[\sum_{i,j} A_{ij} (v D_i \rho)(v D_j \bar{\rho}) - 2 \operatorname{Re}\Big(u \sum_i B_i v D_i \bar{\rho}\Big) + G|u|^2\Big] dx$$

$$+ \int_R \frac{|u|^2}{v} Lv \, dx,$$

from which the conclusions follow as before. The details will be omitted.

Remarks.

5. Note that in the self-adjoint case $b_i = B_i = 0$ $(1 = 1, 2, \cdots, n)$ and $G = 0$, with Γ_2 empty and u real-valued, the operator L is the Euler–Jacobi operator associated with the quadratic functional $M[u]$. In this form Theorem 5.1 is an n–dimensional version of a similar result in the calculus of variations [6].

6. The remarks following Theorem 4.1 also have their counterparts with respect to Theorem 5.1, but these matters will not be pursued any further.

7. Swanson [1] proved the above theorem for non-self-adjoint elliptic operators in an unbounded domain for the special case in which Γ_2 is empty. The present techniques are readily generalized to include unbounded domains provided additional boundary conditions are met at ∞, (see [1]).

8. An application of Green's theorem readily verifies that Theorem 4.1 actually follows from Theorem 5.1. The details for the case when Γ_2 is empty are worked out in [1].

BIBLIOGRAPHY.

[1] C. A. SWANSON, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York and London 1968.

[2] M. PICONE, *Un teorema sulle soluzioni delle equazioni lineari ellitiche autoaggiunte alle derivate parziali del secondo-ordine*, « Atti Rend. Accad. Naz. Lincei », *20*, 213–219 (1911).

[3] F. R. GANTMACHER, *The Theory of Matrices*, Vol. I, Chelsea, New York 1959.

[4] E. HOPF, *A remark on linear elliptic differential equations of the second order*, « Proc. Amer. Math. Soc. », *3*, 791–793 (1952).

[5] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. I, Wiley (Interscience), New York 1953.

[6] O. BOLZA, *Vorlesungen über Variationsrechnung*, Teubner, Berlin 1909.