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Ofelia Teresa Alas

Density and continuous functions

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Density and continuous functions.* Nota di OFELIA TERESA ALAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Studio di alcune proprietà (densità, pseudo–calibro, ecc.) di un tipo particolare di spazi topologici e cenno di possibili applicazioni all'uniforme continuità delle funzioni di variabili reali nei gruppi topologici.

The purpose of this Note is to study certain properties (density, pseudo-caliber, etc.) of a particular type of topological space. These results apply in the study of uniform continuity of real-valued functions in topological groups. They can be regarded as generalizations of some theorems which appear in [2], [4] and [5].

1. INTRODUCTION. DENSITY.

DEFINITION ([3]). The density of a topological space is the smallest cardinal number such that there is a dense subset of this cardinality.

Let q be an infinite cardinal number and T be a set of cardinality $|\mathrm{T}|$ greater or equal to q. For any $t \in \mathrm{T}$, let X_t be a set and σ_t a Hausdorff topology on X_t. On $\mathrm{X} = \prod_{t \in \mathrm{T}} \mathrm{X}_t$ we consider the topology σ generated by the class of the subsets $\prod_{t \in \mathrm{T}} \mathrm{V}_t$ of X, where V_t belongs to σ_t for any $t \in \mathrm{T}$ and $|\{t \in \mathrm{T} \mid \mathrm{V}_t \neq \mathrm{X}_t\}| \leq q$. (A set like this one will be called an elementary open set).

THEOREM 1. *If* $\mathrm{T} = \{0, 1\}^{\mathrm{S}}$, *where* $|\mathrm{S}| \geq q$, *and for any* $t \in \mathrm{T}$, X_t *has density less or equal to* $|\mathrm{S}|^q$, *then* X *has density less or equal to* $|\mathrm{S}|^q$.

Proof. First, we study the set T with the topology generated by the class B of the subsets $\prod_{s \in \mathrm{S}} \mathrm{W}_s$ of T, where $|\{s \in \mathrm{S} \mid \mathrm{W}_s \neq \{0, 1\}\}| \leq q$. Put

(*) Nella seduta del 15 novembre 1969.

$A = \{x \in T \mid |\{s \in S \mid x_s = 1\}| \leq q\}$, where $x = (x_s)_{s \in S}$. It is not difficult to prove that $|A| = |B|$. (For instance, by using the fact that if E is an infinite set, then E and $E \times E$ have the same cardinality). We recall that if E is a set such that $|E| \geq q$, then the set $\{Y \subset E \mid |Y| \leq q\}$ has cardinality less or equal to $|E|^q$. Finally, we prove that $|A| = |B| = |S|^q$.

We fix an element $b \in A$; let H denote the class of the disjoint subsets of B with cardinality less or equal to q; thus, $|H| = |S|^q$. Let us denote by D the set of the functions f of T into A, such that there is $Z \in H$, f is constant in each element of Z and is equal to b outside the union of Z. Thus, $|D| = |S|^q$. Now, for any $t \in T$, let U_t be a dense subset of X_t of cardinality less or equal to $|S|^q$, and g_t be a function of A onto U_t. We put $g : A^T \to \prod_{t \in T} U_t$, where $g((a_t)_{t \in T}) = (g_t(a_t))_{t \in T}$. The set $g(D)$ is dense in X and has cardinality less or equal to $|S|^q$. The proof is completed.

Remark. It is interesting to observe that the intersection of "q" open sets of the topological space T is an open set, thus, the union of "q" closed sets of the topological space T is a closed set. It follows that if P is a subset of T, with $|P| \leq q$, there is a family of disjoint open subsets of T, $(H_p)_{p \in P}$, such that $p \in H_p$ for any $p \in P$. (This can be useful in the last part of the proof of the theorem 1).

Suppose that for any $t \in T$, X_t has at least two points and that m is a cardinal number such that $m^q = m$. Under these conditions we prove the following theorem.

THEOREM 2. *The topological space* X *has density less or equal to* m *if and only if* X_t *has density less or equal to* m *for any* $t \in T$ *and* $|T| \leq 2^m$.

Proof. It is evident that if X has density less or equal to m, then X_t has density less or equal to m for any $t \in T$. If this is true and $|T| \leq 2^m$, by virtue of theorem 1, X has density less or equal to m.

Suppose, now, that X has a dense subset $D = \{(x_t^j)_{t \in T} \mid j \in J\}$, where $|J| \leq m$. For each $t \in T$, let U_t and W_t be two nonempty disjoint open sets of X_t. We consider the function $h : T \to \{0, 1\}^J$, where $h(t) = (y_j(t))_{j \in J}$ and $y_j(t) = 0$ if x_t^j does not belong to U_t and $y_j(t) = 1$ if x_t^j belongs to U_t.

Let $r, p \in T$, $r \neq p$, and consider the elementary open set $\prod_{t \in T} V_t$, where $V_r = U_r$, $V_p = W_p$, and $V_t = X_t$, otherwise. There is $n \in J$, with $(x_t^n)_{t \in T} \in \prod_{t \in T} V_t$; thus, $y_n(r) = 1$ and $y_n(p) = 0$. It follows that h is injective and $|T| \leq 2^m$.

2. CONTINUOUS FUNCTIONS.

In this paragraph we suppose that $|X_t| \geq 2$, for any $t \in T$.

DEFINITION. The pseudo-caliber of a topological space is the smallest cardinal number such that any class of disjoint open sets has cardinality less than this cardinal.

Let m be a cardinal number (not necessarily such that $m^q = m$).

Theorem 3. *If X_t has density less or equal to m^q for any $t \in T$, then X has pseudo-caliber less or equal to the successor of m^q.*

Proof. If $|T| \leq 2^{m^q}$, the density of X is less or equal to m^q and thus the pseudo-caliber of X is less or equal to the successor of m^q.

Suppose that $|T| > 2^{m^q}$ and let $\{U_j \mid j \in J\}$ be a class of disjoint elementary open sets of X; we shall prove that $|J| \leq m^q$. Thus, if for any $j \in J$, $U_j = \prod_{t \in T} V_t^j$, we put $H_j = \{t \in T \mid V_t^j \neq X_t\}$. Let K be a subset of J with cardinality not greater then 2^{m^q} and let T_2 be the union of H_j, when j changes in K. It follows that $|T_2| \leq 2^{m^q}$ and $\{\prod_{t \in T_2} V_t^j \mid j \in K\}$ is a disjoint class of elementary open subsets of $\prod_{t \in T_2} X_t$ (on this last set we consider the topology analogous to the topology σ on X); thus, $|K| \leq m^q$. It follows that $|J| \leq m^q$. The proof is completed.

Notation. Let T' be a subset of T, in the following theorems "$x, y \in X$, $x_t = y_t$, for any $t \in T'$" means that "$x = (x_t)_{t \in T}$, $y = (y_t)_{t \in T}$, and $x_t = y_t$, for any $t \in T'$".

Lemma. *If k is greater or equal to the pseudo-caliber of X and C is an open covering of X, with cardinality less than k, there is a subset T_1 of T, with $|T_1| \leq k$ ($|T_1| < k$ if k is a regular cardinal), such that if $x, y \in X$, $x_t = y_t$ for any $t \in T_1$, then if x belongs to some $Y \in C$, y belongs to the closure of Y.*

Proof. For any $Y \in C$, there is a maximal set H (Y) of elementary open subsets of Y (by virtue of Zorn's theorem). It follows that if $V = \prod_{t \in T} V_t$ belongs to H (Y), then $|\{t \in T \mid V_t \neq X_t\}| \leq q$. We put $L(Y) = \cup\{\{t \in T \mid V_t \neq X_t\} \mid V \in H(Y)\}$ and, since $|H(Y)| < k$, the cardinality of L (Y) is less than k.

Now, let T_1 denote the union of L (Y) when Y changes in C. It follows that $|T_1| \leq k$; furthermore, if k is a regular cardinal then $|T_1| < k$.

We recall that the closure of the union of the set H (Y) is equal to the closure of Y, for any $Y \in C$. This set T_1 verifies the thesis.

Theorem 4. *If k is the pseudo-caliber of X, E is a metrizable space $f: X \to E$ is a continuous function, there is a subset T_2 of T, with $|T_2| \leq k$, such that if $x, y \in X$, $x_t = y_t$ for any $t \in T_2$, then $f(x) = f(y)$.*

Proof. For any natural number $n \geq 1$, there is an open covering B (n) of E, with $|B(n)| < k$, and such that each element has diameter less than $1/n$. Put $C(n) = \{f^{-1}(Z) \mid Z \in B(n)\}$; calculate for C ($n$) the set $T_1(n)$ as in the lemma. Finally put T_2 equal to the union of $T_1(n)$, when $n \geq 1$. Furthermore, if k is a regular cardinal number, then $|T_2| < k$.

Suppose now that E is a Hausdorff space such that each point has a fundamental system of neighborhoods W, with $|W|$ less or equal to q. Under this condition we prove the following theorem.

THEOREM 5. *Suppose that for any* $t \in T$, σ_t *is the discrete topology. If* k *is the pseudo-caliber of* X *and* $f: X \to E$ *is a continuous function, there is a subset* T_1 *of* T, *with* $|T_1| \leq k$, *such that if* $x, y \in X$, $x_t = y_t$ *for any* $t \in T_1$, *then* $f(x) = f(y)$.

Proof. For any $z \in E$ the set $f^{-1}(\{z\})$ is open and closed in X. Put $C = \{f^{-1}(\{z\}) \mid z \in E\}$ and apply the lemma.

THEOREM 6. *Suppose that* $m^q = m$. *If for any* $t \in T$, X_t *has density less or equal to* m *and* $f: X \to E$ *is a continuous function, there is a subset* T_1 *of* T, *with* $|T_1| \leq m$, *such that if* $x, y \in X$, $x_t = y_t$ *for any* $t \in T_1$, *then* $f(x) = f(y)$.

Proof. It is an immediate consequence of theorems 3 and 5.

Remarks. If on X we consider the product topology instead of the topology σ, the analogous of theorems 1, 2 and 3 have been already proved. For instance, Marczewski ([4]) proved the analogue of theorems 1 and 2 for separable spaces; our proof is a generalization of the method developed in [4]. For more details and references see [5] or [2], pages 80, 96, 97 and 98. Ross and Stone proved the analogue of theorem 4 for separable spaces and supposing that E is a second countable Hausdorff space.

To have an idea about the applications of these theorems, as well as of the theorems proved here, to the study of uniform continuity in topological groups see [1].

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