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Limits of double layer potentials

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Analisi matematica. — *Limits of double layer potentials.* Nota di JOSEF KRÁL, presentata [*] dal Corrisp. G. FICHERA.

RIASSUNTO. — In questa Nota si studiano i potenziali di un doppio strato distribuito sulla frontiera di un insieme di Borel nello spazio Euclideo R^m. Si stabiliscono le condizioni necessarie e sufficienti per l'esistenza dei limiti angolari e ordinari dei potenziali con arbitraria densità continua.

Let C be a Borel set in the Euclidean m-space R^m, $m > 1$, and suppose that the boundary B of C is compact. Let $\mathfrak{D}$ denote the class of all infinitely differentiable functions with compact support in R^m. Given $z \in R^m$ we define for any $\psi \in \mathfrak{D}(z) = \{\psi ; \psi \in \mathfrak{D}, z \notin \text{support } \psi\}$

$$W\psi(z) = \int_C \operatorname{grad} \psi(x) \cdot \frac{x-z}{|x-z|^m} \, dx$$

(compare [1] and [2]).

For fixed z,

$$\psi \to W\psi(z) \tag{1}$$

represents a distribution on $R^m \setminus \{z\}$ whose support is contained in B. Necessary and sufficient geometric condition for the functional (1) to be continuous with respect to the uniform convergence in $\mathfrak{D}(z)$ was established in lemma 2.2 in [1] and may be expressed by (2) below. Let us briefly recall the necessary notation. Put $\Gamma = \{\theta ; \theta \in R^m, |\theta| = 1\}$ and, for each $\theta \in \Gamma$, consider the half-line $H^\theta = \{z + r\theta ; r > 0\}$. We shall write $\mathfrak{S}(z) = \{H^\theta ; \theta \in \Gamma\}$. A point $y \in H^\theta$ is termed a hit of H^θ on C provided each neighborhood of y meets both $H^\theta \cap C$ and $H^\theta \setminus C$ in a set of positive linear measure. The total number $n(\theta, z)$ of all the hits of H^θ on C is a Baire function of the variable $\theta \in \Gamma$ and we may define

$$v(z) = \int_\Gamma n(\theta, z) \, dH_{m-1}(\theta),$$

where H_{m-1} is the Hausdorff $(m-1)$-measure. Then

$$v(z) < \infty \tag{2}$$

(*) Nella seduta del 10 gennaio 1970.

(1) J. KRÁL, *The Fredholm method in potential theory*, *Transactions of the American Mathematical Society*, *125*, 511–547 (1966).

(2) JU. D. VURAGO and V. G. MAZJA, *Some questions in potential theory and function theory for regions with irregular boundaries* (*Russian*), Zapiski naučnych seminarov LOMI, *3*, 5–86 (1967).

is a necessary and sufficient condition for the existence of a finite signed Borel measure ν_z such that

$$W\psi(z) = \int \psi \, d\nu_z, \qquad \psi \in \mathfrak{D}(z). \tag{3}$$

The measure ν_z is uniquely determined by (3) and the additional assumption $\nu_z(\{z\}) = 0$. The support of ν_z is contained in B and $v(z)$ equals the total variation of ν_z. Assuming (2) we may thus define

$$Wf(z) = \int f \, d\nu_z \tag{4}$$

for each bounded Baire function f on B. The quantity (4) may be interpreted as the value at z of the potential of the double layer with density f. Indeed, if the boundary B of C is formed by a sufficiently smooth hypersurface, then $Wf(z)$ reduces to

$$\int_B f(y) \frac{n(y)\cdot(y-z)}{|y-z|^m} \, dH_{m-1}(y),$$

where $n(y)$ stands for the exterior normal at $y \in B$. For further investigation of the double layer potential it appears natural to adopt the following assumption (A):

(A) *There exists an* $(m+1)$*-tuple of points* $z^1, \cdots, z^{m+1} \in R^m$ *in general position (i.e., not situated on a single hyperplane) such that* $v(z^j) < \infty$ *for all* $j = 1, \cdots, m+1$.

In what follows we always assume (A). Then $v(z) < \infty$ for all $z \in R^m \setminus B$ and, moreover, $\sup v(M) < \infty$ for each set $M \subset R^m \setminus B$ having a positive distance from B (see 2.9 and 2.10 in [1]). Let $\mathfrak{B}$ denote the Banach space of all bounded Baire functions on B equipped with the supremum norm. $\mathfrak{C}$ will stand for the subspace of all continuous functions in $\mathfrak{B}$. For each $f \in \mathfrak{B}$, $Wf(z)$ is defined for all $z \notin B$ and represents a harmonic function in $R^m \setminus B$. We thus arrive at a continuous map $W: f \to Wf$ of $\mathfrak{B}$ into the space of all harmonic functions in $R^m \setminus B$ endowed with the topology of compact convergence. Even if $f \in \mathfrak{C}$, Wf need not, in general, possess angular limits at points of B. We now fix a point $y \in B$ and are going to investigate necessary and sufficient conditions on C guaranteeing the existence of angular limits of Wf at y for each $f \in \mathfrak{C}$. For $M \subset R^m$ we denote by $\bar{M}$ the closure of M. If $y \in R^m$ then $\mathrm{contg}_M y$ will stand for the contingent of M at y which consists of all the half-lines $\{y + r\theta ; r > 0\} \in \mathfrak{S}(y)$ for which there is a sequence of points $z^n \in M \setminus \{y\}$ tending to y such that $\lim_n (z^n - y)/|z^n - y| = \theta$. $\Omega_r(z)$ will denote the open ball of radius r and center z and $d(z)$ will denote the m–density of C at z defined by

$$d(z) = \lim_{r \to 0+} \frac{\text{volume}(\Omega_r(z) \cap C)}{\text{volume}\, \Omega_r(z)} \tag{5}$$

for those $z \in R^m$ for which the limit (5) exists. (According to 2.7 in [1], $d(z)$ is available whenever (2) holds). Finally, put $C_\alpha = \{z ; d(z) = \alpha\}$ for $\alpha = 0, 1, 1/2$. Clearly, $C_{1/2} \subset B$, $C_1 \subset \bar{C}$, $R^m \setminus \bar{C} \subset C_0$. Now we are in a position to announce

THEOREM 1. *Let $y \in B$ and suppose that there is a $\theta \in \Gamma$ such that neither $\{y + r\theta ; r > 0\}$ nor $\{y - r\theta ; r > 0\}$ belong to* $\text{contg}_B y$. *If*

$$\limsup_{\substack{z \to y \\ z \in H}} |Wf(z)| < \infty$$

for each $H \in \mathcal{S}(y) \setminus \text{contg}_B y$ and each $f \in \mathcal{C}$, then

$$v(y) < \infty \tag{6}$$

and

$$\sup_{r>0} \frac{H_{m-1}(\Omega_r(y) \cap C_{1/2})}{r^{m-1}} < \infty. \tag{7}$$

The proof of this theorem is based on the principle of uniform boundedness and on reasonings similar to those used for the proof of the estimate (2.17) in [1].

The conditions (6), (7) are not only necessary, but also sufficient for the existence at y of angular limits of the double layer potential with any continuous density. This follows from

THEOREM 2. *Let $S \subset R^m \setminus B$, $y \in \bar{S} \cap B$ and suppose that*

$$(\text{contg}_{C_{1/2}} y) \cap \text{contg}_S y = \varnothing.$$

If (6), (7) *hold then*

$$\limsup_{\substack{z \to y \\ z \in S}} |Wf(z)| < \infty$$

for each $f \in \mathfrak{B}$. If, besides that, $\text{contg}_S y$ *reduces to a single half-line, then there are $r > 0$ and i (= 0 or 1) such that*

$$\Omega_r(y) \cap S \subset C_i$$

and

$$\lim_{\substack{z \to y \\ z \in S}} Wf(z) = Wf(y) + f(y)\, H_{m-1}(\Gamma)\,(i - d(y))$$

or each $f \in \mathfrak{B}$ that is continuous at y.

A refinement of the argument used for the proof of theorem 2.15 in [1] yields also the following theorem concerning ordinary limits of double layer potentials.

THEOREM 3. *Let* $y \in B$. *If*

$$\limsup_{\substack{z \to y \\ z \notin B}} |Wf(z)| < \infty \tag{8}$$

for each $f \in \mathcal{C}$, *then*

$$\limsup_{\substack{x \to y \\ x \in B}} v(x) < \infty . \tag{9}$$

Conversely, if (9) *holds, then* $v(y) < \infty$ *and* (8) *is valid for each* $f \in \mathfrak{B}$; *if, besides that,* f *is continuous at* y, *then the limits*

$$L_i = \lim_{\substack{z \to y \\ z \in C_i}} Wf(z)$$

exist for $i = 0, 1$ (*in the case* $y \notin \bar{C}_i$ *we set* $L_i = (-1)^{i-1} f(y) H_{m-1}(\Gamma) + Wf(y)$ *ex definitione*) *and*

$$L_1 - L_0 = H_{m-1}(\Gamma) f(y) \quad , \quad L_1 + L_0 = 2\, Wf(y) + f(y)\, H_{m-1}(\Gamma)\,(1 - 2\, d(y)) .$$

Remark. – There is an extensive literature dealing with double layer potentials and their applications. References to the work of G. C. Evans, G. Fichera, R. Leis, E. R. C. Miles, C. Müller and others may be found in S. Dümmel's article (3). The corresponding bibliography and further details will be given elsewhere.

(3) S. DÜMMEL, *Einige Eigenschaften von k-dimensionalen λ-Potentialen der einfachen und der doppelten Belegung*, « Atti della Accademia Nazionale dei Lincei, Memorie », ser. VIII, vol. VII, 173–201 (1965).