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On a non—linear mixed problem for the Navier-Stokes equations. Nota I

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Analisi matematica. — *On a non-linear mixed problem for the Navier–Stokes equations.* Nota I di GIOVANNI PROUSE (*), presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si considera, per le equazioni di Navier–Stokes, un problema misto con condizioni al contorno non lineari dedotte da un problema fisico concreto e si dimostra che, in ipotesi di regolarità di tipo classico, tale problema ammette al più una soluzione. Teoremi di esistenza ed unicità per soluzioni generalizzate opportunamente definite vengono dimostrati nelle successive Note II e III.

1. – The equations of Navier-Stokes have been the object of a great deal of research work, both theoretical and experimental, and constitute one of the most interesting examples of non-linear equations of Mathematical Physics.

Many mixed problems (according to Hadamard) can obviously be considered for these equations, assigning initial conditions and appropriate boundary conditions.

The formulation of these boundary conditions and their influence on the solution of the problem are well focused by an " energy relation " which we shall now obtain in a purely formal way.

Let Ω be a bounded open set in an Euclidean m-dimensional space $x = \{x_1, \cdots, x_m\}$ and let Γ be its boundary. If μ denotes the viscosity coefficient, $\vec{u}(x, t) = \{u_1(x, t), \cdots, u_m(x, t)\}$ the velocity, $p(x, t)$ the pressure, $\vec{f}(x, t) = \{f_1(x, t), \cdots, f_m(x, t)\}$ the external force, the equations of Navier–Stokes which govern the motion in Ω of the fluid (which is assumed to be incompressible and of unit density) are

$$
(1.1) \qquad \begin{cases} \dfrac{\partial u_j}{\partial t} - \mu \Delta u_j + \displaystyle\sum_{k=1}^{m} u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial p}{\partial x_j} = f_j & (j = 1, \cdots, m) \\ \displaystyle\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_k} = 0 . \end{cases}
$$

Equations (1.1) can also be written, when $m \leq 3$, in the following vector form

$$
(1.2) \qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\operatorname{rot} \vec{u}) \wedge \vec{u} + \operatorname{grad}\left(\dfrac{|\vec{u}|^2}{2} + p\right) - \mu \Delta \vec{u} = \vec{f} \\ \operatorname{div} \vec{u} = 0 . \end{cases}
$$

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In the present note we shall systematically utilize, for the sake of simplicity, this vector form, although the results obtained are valid also in the case of any m.

Let us consider the scalar product of the first of (1.2) by $\vec{u}$ and integrate over Ω; observing that $(\operatorname{rot}\vec{u}) \wedge \vec{u} \times \vec{u} = 0$, we obtain

$$\int_{\Omega} \left(\frac{\partial \vec{u}}{\partial t} + \operatorname{grad}\left(\frac{|\vec{u}|^2}{2} + p \right) - \mu \Delta \vec{u} \right) \times \vec{u} \, d\Omega = \int_{\Omega} \vec{f} \times \vec{u} \, d\Omega \tag{1.3}$$

and, as can easily be seen,

$$\operatorname{grad} \frac{|\vec{u}|^2}{2} \times \vec{u} = \sum_{j,k=1}^{m} u_j \frac{\partial u_k}{\partial x_j} u_k .$$

On the other hand, denoting by $\vec{n}$ the exterior normal to Γ, we have, by Green's formula,

$$\int_{\Omega} \sum_{j,k=1}^{m} u_j \frac{\partial u_k}{\partial x_j} u_k \, d\Omega = - \int_{\Omega} \sum_{j,k=1}^{m} u_k \frac{\partial}{\partial x_j} (u_j u_k) \, d\Omega + \int_{\Gamma} \sum_{j,k=1}^{m} u_j u_k^2 \cos \vec{n} x \, d\Gamma =$$

$$= - \int_{\Omega} \sum_{j,k=1}^{m} u_j \frac{\partial u_k}{\partial x_j} u_k \, d\Omega - \int_{\Omega} \sum_{j,k=1}^{m} u_k^2 \frac{\partial u_j}{\partial x_j} \, d\Omega + \int_{\Gamma} |\vec{u}|^2 \sum_{j=1}^{m} u_j \cos \vec{n} x_j \, d\Gamma .$$

Hence, as $\operatorname{div} \vec{u} = 0$.

$$\int_{\Omega} \operatorname{grad} \frac{|\vec{u}|^2}{2} \times \vec{u} \, d\Omega = \frac{1}{2} \int_{\Gamma} |\vec{u}|^2 \vec{u} \times \vec{n} \, d\Omega . \tag{1.4}$$

Setting $[\vec{u}, \vec{v}] = \int_{\Omega} \sum_{j,k=1}^{m} \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k} \, d\Omega$ (so that $[\vec{u}, \vec{u}]$ is a seminorm on $H^1(\Omega)$), it results, analogously,

$$\int_{\Omega} \Delta \vec{u} \times \vec{u} \, d\Omega = - [\vec{u}, \vec{u}] + \int_{\Gamma} \vec{u} \times \frac{\partial \vec{u}}{\partial \vec{n}} \, d\Gamma \tag{1.5}$$

and, moreover, since $\operatorname{div} \vec{u} = 0$,

$$\int_{\Omega} \operatorname{grad} p \times \vec{u} \, d\Omega = \int_{\Gamma} p \vec{u} \times \vec{n} \, d\Gamma . \tag{1.6}$$

Introducing the usual notations $\vec{u}(t) = \{\vec{u}(x, t) ; x \in \Omega\}$, $\vec{u}'(t) = \left\{ \frac{\partial \vec{u}(x,t)}{\partial t} ; x \in \Omega \right\}$, $\Delta \vec{u}(t) = \{\Delta \vec{u}(x, t) ; x \in \Omega\}$, $\vec{f}(t) = \{\vec{f}(x, t) ; x \in \Omega\}$, $p(t) = \{p(x, t) ; x \in \Omega\}$, $(\vec{u}, \vec{v})_{L^2} = \int_{\Omega} \sum_{j=1}^{m} u_j v_j \, d\Omega$, $\| \vec{u} \|_{L^2}^2 = (\vec{u}, \vec{u})_{L^2}$ it follows from (1.3), (1.4), (1.5), (1.6), that

$$(1.7) \qquad \frac{1}{2}\frac{d}{dt}\|\vec{u}(t)\|^2_{L^2} + \mu\,[\vec{u}(t)\,,\vec{u}(t)] = (\vec{f}(t)\,,\vec{u}(t))_{L^2} + \mu\int_\Gamma \vec{u}(x\,,t)\times$$

$$\times\frac{\partial\vec{u}(x\,,t)}{\partial\vec{n}}\,d\Gamma - \int_\Gamma\left(\frac{1}{2}|\vec{u}(x\,,t)|^2 + p(x\,,t)\right)\vec{u}(x\,,t)\times\vec{n}\,d\Gamma\,.$$

Observe that (having assumed that the density is equal to 1) the quantity $\frac{1}{2}|\vec{u}|^2 + p$ represents the "energy", sum of the kinetic and of the piezometric energies, of the fluid; it has therefore a precise physical interpretation and has a fundamental role for instance in Bernoulli's classical theorem. Observe moreover that the non-linearity of the equations of Navier-Stokes appears in the last integral on the right hand side of (1.7).

It is evident that, if we assign the values of $\vec{u}$ on Γ (Dirichlet boundary conditions) the "energy relation" (1.7) becomes particularly simple; assuming, in particular, that $\vec{u}\,|_\Gamma = 0$, we have

$$(1.8) \qquad \frac{1}{2}\frac{d}{dt}\|\vec{u}(t)\|^2_{L^2} + \mu\,[\vec{u}(t)\,,\vec{u}(t)] = (\vec{f}(t)\,,\vec{u}(t))_{L^2}$$

and it is therefore possible to obtain apriori estimates for the solutions. A relation similar to (1.8) from which apriori estimates can be deduced holds also if $\vec{u}\,|_\Gamma \neq 0$ (see Ladyzenskaja [1]).

In many cases of physical interest the term $\int_\Gamma \vec{u}\times\frac{\partial\vec{u}}{\partial\vec{n}}\,d\Gamma$ vanishes. This happens, for instance, if Ω is a cylinder with axis parallel to x_1, and initial and final sections orthogonal to x_1, and if we assume that on Γ the velocity $\vec{u}$ is parallel to $\vec{n}$.

Most of the mathematical research has so far been dedicated to the study of the mixed problem with Dirichlet boundary conditions: the existence (in the large) of a solution of this problem, intended in an appropriate sense, has been proved by Hopf [2] utilizing essentially relation (1.8). The problem of the uniqueness of such a solution has not yet been completely solved and it is doubtful that a uniqueness theorem holds (except for $m = 2$) in the functional class introduced in the existence theorem; see, on this subject a recent example given by Ladyzenskaja [3]. On the other hand, in those classes for which the uniqueness theorem has been proved, the corresponding existence theorem holds only for sufficiently small t (see, for example, Kieslev and Ladyzenskaja [4], Shinbrot and Kaniel [5], Prouse [6]). For an extensive list of references and a detailed report of the results obtained we refer to the book [1] by Ladyzenskaja.

A mixed problem with boundary conditions not of Dirichlet type, and moreover non linear, has been recently suggested to me by G. Noseda of the

Istituto di Idraulica of the Politecnico of Milan. Consider the motion of a viscous incompressible fluid in a cylindrical tube with permeable wall and surrounded itself by the same fluid; there is then a flow through the wall with velocity, directed orthogonally to the wall, which can be estimated empirically to be proportional to the square root of the jump of pressure. This problem is actually encountered, for example, when studying the motion of the blood in artificial arteries which, at least for a certain time, are permeable.

In order that the problem be mathematically well posed it is necessary to complete the data by assigning appropriate conditions on the initial and final sections of the tube. It seems to me interesting to observe that the problem appears well posed and can be solved (at least in the case $m = 2$) when on these sections we assign the quantity $\frac{1}{2} |\vec{u}|^2 + p$ already recalled above (thus introducing a further non-linearity in the problem). After having given in § 2 a uniqueness theorem which holds under classical smoothness assumptions on the solution, we shall in fact prove in the two following notes, in the case $m = 2$, a uniqueness theorem and an existence theorem, for sufficiently small t, of the solution in a functional class in which both theorems hold simultaneously.

We shall, finally, mention some results which can be obtained for problems slightly different from the one just considered.

2. Let $\{\vec{u}(x, t), p(x, t)\}$, $\{\vec{v}(x, t), q(x, t)\}$ be two solutions of the system (1.2), in the classical sense. Assume that, when $t = 0$,

$$\vec{u}(x, 0) = \vec{v}(x, 0) \tag{2.1}$$

and moreover that, denoting by Γ_1, Γ_2, Γ_3 respectively the initial section, the wall and the final section of the tube (which we shall suppose is cylindrical, with axis parallel to x_1),

$$\frac{1}{2} |\vec{u}|^2 + p = \frac{1}{2} |\vec{v}|^2 + q \qquad \text{on} \quad \Gamma_1 \cup \Gamma_3, \tag{2.2}$$

$$\vec{u} \times \vec{\tau} = \vec{v} \times \vec{\tau} = 0 \qquad \text{on} \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma, \tag{2.3}$$

$$p - \beta \vec{u} \times \vec{n} \, |\vec{u}| = q - \beta \vec{v} \times \vec{n} \, |\vec{v}| = 0 \qquad \text{on} \quad \Gamma_2, \tag{2.4}$$

$\vec{n}$ being the normal unit vector to Γ and $\vec{\tau}$ any unit vector tangent to Γ.

Relation (2.2) expresses the condition that the two flows have, on the initial and final sections, the same energy; (2.4) interprets (according to what has been mentioned in § 1) the relation between the pressure jump and the velocity of the fluid flowing through Γ_2, assuming that the external pressure is zero and denoting by $\beta \geq 0$ the permeability coefficient. Finally (2.3) imposes (according to the well-known theory of the limit layer) that the components of the velocity parallel to $\vec{\tau}$ vanish along Γ_2 and, moreover, that on the initial and final sections (which, as we have assumed, are orthogonal to x_1) the velocity is parallel to the axis of the tube.

Let us now prove that, under the assumptions made above, $\vec{u}(x,t) = \vec{v}(x,t)$, i.e. that *the flow is uniquely determined if we assign the initial velocity and the boundary conditions indicated in* § 1.

Setting $\vec{w} = \vec{u} - \vec{v}$, the vector $\vec{w}$ obviously satisfies the system

$$(2.5)\quad \begin{cases} \dfrac{\partial \vec{w}}{\partial t} + (\operatorname{rot} \vec{u}) \wedge \vec{u} - (\operatorname{rot} \vec{v}) \wedge \vec{v} + \operatorname{grad}\left(\dfrac{|\vec{u}|^2 - |\vec{v}|^2}{2} + p - q\right) - \mu\, \Delta \vec{w} = 0 \\ \operatorname{div} \vec{w} = 0 \end{cases}$$

and, $\vec{u}$ and $\vec{v}$ being "regular" solutions, we may assume that $\vec{w}(x,t)$ has continuous first derivatives with respect to all variables.

From (2.5) we obtain analogously to (1.7), by scalar multiplication by $\vec{w}$ and integration over Ω, bearing in mind (2.3),

$$(2.6)\quad \frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2_{L^2} + \mu\,[\vec{w}(t), \vec{w}(t)] + \int_\Omega [(\operatorname{rot}\vec{u}(x,t)) \wedge \vec{u}(x,t) - (\operatorname{rot}\vec{v}(x,t)) \wedge \vec{v}(x,t)] \times \vec{w}(x,t)\, d\Omega + \int_\Gamma \left[\frac{1}{2}|\vec{u}(x,t)|^2 + p(x,t) - \frac{1}{2}|\vec{v}(x,t)|^2 - q(x,t)\right] \vec{w}(x,t) \times \vec{n}\, d\Gamma = 0.$$

By (2.2), (2.3), (2.4) it follows from (2.6) that

$$(2.7)\quad \frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2_{L^2} + \mu\,[\vec{w}(t), \vec{w}(t)] + \int_\Omega [(\operatorname{rot}\vec{u}(x,t)) \wedge \vec{w}(x,t) \times \vec{w}(x,t) + (\operatorname{rot}\vec{w}(x,t)) \wedge \vec{v}(x,t) \times \vec{w}(x,t)]\, d\Omega + \int_{\Gamma_2} \left[\frac{1}{2}|\vec{u}(x,t)|^2 + \beta(x,t)\,|\vec{u}(x,t)|\,\vec{u}(x,t) \times \vec{n} - \frac{1}{2}|\vec{v}(x,t)|^2 - \beta(x,t)\,|\vec{v}(x,t)|\,\vec{v}(x,t) \times \vec{n}\right] \vec{w}(x,t) \times \vec{n}\, d\Gamma_2 .$$

Since $(\operatorname{rot}\vec{u}) \wedge \vec{w} \times \vec{w} = 0$ and moreover (as $|\xi|\xi$ is increasing and $\beta \geq 0$) $\int_{\Gamma_2} \beta(|\vec{u}|\vec{u} \times \vec{n} - |\vec{v}|\vec{v} \times \vec{n})\vec{w} \times \vec{n}\, d\Gamma_2 = \int_{\Gamma_2} \beta(|\vec{u} \times \vec{n}|\,\vec{u} \times \vec{n} - |\vec{v} \times \vec{n}| \cdot$

$\cdot \vec{v} \times \vec{n}) \vec{w} \times \vec{n} \, d\Gamma_2 \geq 0$, we also obtain

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} \| \vec{w}(t) \|_{L^2}^2 + \mu [\vec{w}(t), \vec{w}(t)] \leq \left| \int_\Omega (\operatorname{rot} \vec{w}(x,t)) \wedge \vec{v}(x,t) \times \vec{w}(x,t) \, d\Omega \right| +$$

$$+ \frac{1}{2} \left| \int_{\Gamma_2} [|\vec{u}(x,t)|^2 - |\vec{v}(x,t)|^2] \, \vec{w}(x,t) \times \vec{n} \, d\Gamma_2 \leq c_1 \int_\Omega | \operatorname{rot} \vec{w}(x,t) | \cdot \right.$$

$$\cdot |\vec{w}(x,t)| \, d\Omega + \frac{1}{2} \int_{\Gamma_2} (|\vec{u}(x,t)| + |\vec{v}(x,t)|) \left| |\vec{u}(x,t)| - |\vec{v}(x,t)| \right| |\vec{w}(x,t)| \, d\Gamma_2 .$$

Bearing in mind the relation $|ab| \leq \varepsilon |a|^2 + k_\varepsilon |b|^2$ there exists, on the other hand, a constant k_μ such that

$$(2.9) \quad \int_\Omega |\operatorname{rot} \vec{w}(x,t)| \, |\vec{w}(x,t)| \, d\Omega \leq \int_\Omega \left\{ \frac{\mu}{2} \sum_{j,k=1}^m \left(\frac{\partial w_j(x,t)}{\partial x_k} \right)^2 + \right.$$

$$\left. + k_\mu \sum_{j=1}^m w_j^2(x,t) \right\} d\Omega = \frac{\mu}{2} [\vec{w}(t), \vec{w}(t)] + k_\mu \| \vec{w}(t) \|_{L^2}^2$$

Moreover,

$$(2.10) \quad \int_{\Gamma_2} (|\vec{u}(x,t)| + |\vec{v}(x,t)|) \left| |\vec{u}(x,t)| - |\vec{v}(x,t)| \right| |\vec{w}(x,t)| \, d\Gamma_2 \leq$$

$$\leq \int_{\Gamma_2} (|\vec{u}(x,t)| + |\vec{v}(x,t)|) \, |\vec{w}(x,t)|^2 \, d\Gamma_2 \leq c_2 \int_{\Gamma_2} |\vec{w}(x,t)|^2 \, d\Gamma_2 .$$

Introducing (2.9), (2.10) into (2.8) we obtain

$$(2.11) \quad \frac{1}{2} \frac{d}{dt} \| \vec{w}(t) \|_{L^2}^2 + \frac{\mu}{2} [\vec{w}(t), \vec{w}(t)] \leq c_3 \| \vec{w}(t) \|_{L^2}^2 + c_4 \int_{\Gamma_2} |\vec{w}(x,t)|^2 \, d\Gamma_2 .$$

As $\vec{u}(x,t), \vec{v}(x,t)$ are two fixed vectors which, as mentioned above, have continuous first derivatives, there exists a constant c_5, depending on $\vec{u}$ and $\vec{v}$ but not on t, such that

$$\int_{\Gamma_2} |\vec{w}(x,t)|^2 \, d\Gamma_2 \leq c_5 \int_\Omega |\vec{w}(x,t)|^2 \, d\Omega = c_5 \| \vec{w}(t) \|_{L^2}^2 .$$

Hence

$$(2.12) \quad \frac{d}{dt} \| \vec{w}(t) \|_{L^2}^2 + \frac{\mu}{2} [\vec{w}(t), \vec{w}(t)] \leq c_6 \| \vec{w}(t) \|_{L^2}^2 .$$

Since, by (2.1), $\| w(0) \|_{L^2} = 0$, it follows that $\vec{w}(t) = 0$. The uniqueness theorem is therefore proved.

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