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Hans Lewy

On a refinement of Evans' law in potential theory

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On a refinement of Evans' law in potential theory.* Nota di HANS LEWY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si considerino una misura μ a sostegno compatto S contenuto nell'asse delle x, il suo potenziale logaritmico $v(x) = \int_S -\log|x-y|\, d\mu(y)$, ed una funzione $f(x)$ avente derivata continua e tale che $f(x) = v(x)$ per $x \in S$ e $f(x) \leq v(x)$ per $x \notin S$. Si dimostra che anche $v(x)$ ha derivata continua. Si indicano alcune conseguenze di questo teorema per le funzioni armoniche di due variabili.

INTRODUCTION.

Given a (smooth) bounded domain Ω of $\mathbf{R}^n$, $n \geq 2$, and a continuous real function $f(x)$, $x \in \Omega$, which is negative on $\partial\Omega$ (or negative outside a compact set $\subset \Omega$) there is a smallest continuous superharmonic $u(x)$ with $u(x) \geq f(x)$ in Ω, $u(x) \geq 0$ on $\partial\Omega$. If $f(x)$ is sufficiently smooth, e.g. if $|\Delta f(x)|$ is bounded, then $u(x)$ has Holder continuous derivatives [1], [2], [3]. Suppose next that $f(x)$ is a continuous function, defined *only* on an $(n-1)$ dimensional plane section $\mathbf{R}^{n-1} \cap \Omega$, and negative on $\partial\Omega \cap \mathbf{R}^{n-1}$. Let $f^+(x)$ be the non-negative part of $f(x)$ and let $(\mathrm{T}f)(x)$ be a continuous harmonic which vanishes on $\partial\Omega$ and equals $f^+(x)$ on $\mathbf{R}^{n-1} \cap \Omega$. If $u(x)$ is the smallest continuous superharmonic, ≥ 0 on $\partial\Omega$ and $\geq f^+(x)$ on $\mathbf{R}^{n-1} \cap \Omega$, then $u(x) \geq (\mathrm{T}f)(x)$ and, conversely, the existence of a continuous smallest superharmonic $u(x) \geq (\mathrm{T}f)(x)$ implies that of a smallest continuous superharmonic u with $u(x) \geq f^+(x)$ on $\mathbf{R}^{n-1} \cap \Omega$.

(*) Nella seduta del 10 gennaio 1970.

But this question arises: Suppose $f(x)$, $x \in \mathbf{R}^{n-1} \cap \Omega$, is smooth, does this imply that the *restriction* of $u(x)$ to $\mathbf{R}^{n-1} \cap \Omega$ is smooth?

In this paper the affirmative answer is given for $n = 2$: If $f(x) \in C^1$, $x \in \mathbf{R}^1 \cap \Omega$, then $u(x) \in C^1$, $x \in \mathbf{R}^1 \cap \Omega$.

§ 1. *Existence of one-sided derivatives for the logarithmic potential.*

A famous theorem of Evans states that the potential of a positive distribution of mass, if continuous on the restriction to the support of the mass (smallest compact set containing all the mass) then it is continuous throughout. The first theorem to be proved may be considered a refinement of Evans' law and reads as follows:

THEOREM 1.1. *Let* $\mu(x)$, $x \in \mathbf{R}^1$, *be a positive measure of compact support* S *and*

$$u(x) = \int_S -\log|x-y| \, d\mu(y), \qquad x \in \mathbf{R}^1$$

its potential. Let $I = (A, B)$ *be an open interval containing* S. *Suppose there exists a real function* $f(x)$ *which is* $C^1(I)$ *and such that* $f(x) \leq u(x)$, $x \in I$, *and* $f(x) = u(x)$ *on* S, *then* $u(x)$ *is* $C^1(I)$.

The complement of S in I is denoted by $\complement S$; it consists of an at most countable number of disjoint component intervals (a_i, b_i).

LEMMA 1.1. *Under the assumptions of Theorem 1.1, let* $x_0 \in I$ *be the right endpoint of a component interval of* $\complement S$. *Then* $u(x)$ *has a derivative from the left, denoted by* $\frac{du}{dx^-}(x_0)$, *and* $\frac{du}{dx^-}(x_0) \leq \frac{df}{dx}(x_0)$.

Remark. The following proof requires of $f(x)$ only that it be continuous on I and have a derivative at x_0.

Proof of Lemma 1.1. Put

$$S_1 = \{x : x < x_0, x \in S\}, \quad S_2 = \{x : x \geq x_0, x \in S\},$$

$$u_j(x) = \int_{S_j} -\log|x-y| \, d\mu(y), \qquad j = 1, 2.$$

Obviously, $\frac{du_1}{dx}(x_0)$ exists. For $x < x_0$,

$$\frac{u_2(x) - u_2(x_0)}{x - x_0} = \int_{S_2} \frac{-\log|y-x| + \log|y-x_0|}{x - x_0} \, d\mu(y).$$

Put $s = \frac{x_0 - x}{y - x_0}$. Since $u(x)$ is continuous on S, $\mu(x_0) = 0$ and the last integral need be taken only over $S_2 - \{x_0\}$. This makes s finite and > 0 in the integrand. On $S_2 - \{x_0\}$

$$(y - x_0) \cdot \frac{-\log|y-x| + \log|y-x_0|}{x - x_0} = s^{-1} \log(s+1)$$
$$= s^{-1} \int_1^{1+s} \frac{dt}{t}.$$

For $x \uparrow x_0$, $s \downarrow 0$ and $s^{-1} \log (1 + s) \uparrow 1$. By Beppo Levi's theorem

$$\frac{u_2(x) - u_2(x_0)}{x - x_0} \xrightarrow[x \uparrow x_0]{} \int_{S_2} \frac{d\mu(y)}{y - x_0}$$

increasingly. Thus for $x \uparrow x_0$,

$$\frac{u(x) - u(x_0)}{x - x_0} = \frac{u_1(x) - u_1(x_0)}{x - x_0} + \frac{u_2(x) - u_2(x_0)}{x - x_0} \to \frac{du_1}{dx}(x_0) + \int_{S_2} \frac{d\mu}{y - x_0}.$$

But left hand is by hypothesis $\leq \dfrac{f(x) - f(x_0)}{x - x_0}$. Hence

$$f'(x_0) \geq \int_{S_1} \frac{d\mu(y)}{y - x_0} + \int_{S_2} \frac{d\mu(y)}{y - x_0}. \tag{1.1}$$

In particular $\displaystyle\int_{S_2} \frac{d\mu(y)}{y - x_0} < \infty$. Lemma 1.1 is proved.

A reversal of the positive direction of the x–axis does not affect the correspondence of the points P of I with $u(\mathrm{P})$ or $f(\mathrm{P})$, but changes a left endpoint of a component interval into a right endpoint, and changes $f'(\mathrm{P})$ into $-f'(\mathrm{P})$ and thus changes the sign of the inequality (1.1). Thus we have

LEMMA 1.2. *Under the assumptions of Theorem 1.1, let $x_0 \in \mathrm{I}$ be the left endpoint of a component interval of* $\complement\mathrm{S}$. *Then $u(x)$ has a derivative $\dfrac{du}{dx^+}(x_0)$ from the right and $\dfrac{du}{dx^+}(x_0) \geq \dfrac{df}{dx}(x_0)$.*

Let x_0, $x_0' > x_0$ be two points of the same component interval (a, b) of $\complement\mathrm{S}$, and put $\mathrm{S}_1 = \{x : x \in \mathrm{S}, x < x_0\}$, $\mathrm{S}_2 = \{x : x \in \mathrm{S}, x > x_0\}$. We then have

$$\int_{S_1} \frac{d\mu(y)}{y - x_0} \leq \int_{S_1} \frac{d\mu(y)}{y - x_0'} \quad , \quad \int_{S_2} \frac{d\mu(y)}{y - x_0} \leq \int_{S_2} \frac{d\mu(y)}{y - x_0'} ,$$

with equality excluded in at least one of these inequalities. Addition yields

$$\frac{du}{dx}(x_0) < \frac{du}{dx}(x_0'). \tag{1.2}$$

Since du/dx is thus continuous and monotone on the open interval (a, b) we have

$$\frac{du}{dx^+}(a) = \lim_{x \downarrow a} \frac{du}{dx} \quad , \quad \frac{du}{dx^-}(b) = \lim_{x \uparrow b} \frac{du}{dx}.$$

By Lemmas 1.2 and 1.1 and formula (1.2), if a and b are *both* in S,

$$f'(a) \leq \frac{du}{dx^+}(a) < \frac{du}{dx^-}(b) \leq f'(b) \tag{1.3}$$

so that

$$f'(a) < f'(b). \tag{1.4}$$

The total variation of $\mathrm{d}u/\mathrm{d}x$ in (a, b) is not in excess of $f'(b) - f'(a)$, which is certainly not greater than the total variation of $f'(x)$ in (a, b). Another consequence of (1.4), incidentally, is that if $f(x)$ is concave, S consists of exactly one closed interval as $\complement$S cannot possess component intervals (a, b) with both a and b in S.

LEMMA 1.3. *Let $x_0 \in \mathrm{S}$ be such that there exists a sequence of points $x_i \in \mathrm{S}$ with $x_i > x_0$ (resp. $x_i < x_0$) and $\lim_{i \to \infty} x_i = x_0$. Then $\frac{\mathrm{d}u}{\mathrm{d}x^+}(x_0)$ $\left(resp.\ \frac{\mathrm{d}u}{\mathrm{d}x^-}(x_0)\right)$ exists and equals $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$.*

Proof. We only need prove the case $x_i > x_0$, $x_i \in \mathrm{S}$. Two possibilities occur: There is an interval $[x_0, \mathrm{X}]$, $\mathrm{X} > x_0$, which belongs to S, and consequently $u(x) = f(x)$ so that the assertion of Lemma 1.3 becomes trivial. Or x_0 is limit of component intervals (a_i, b_i) of $\complement$S, with $b_i > a_i > x_0$. If $x \in (a_i, b_i)$, the monotonicity of $\mathrm{d}u/\mathrm{d}x$ in (a_i, b_i) together with (1.3) yield

$$(x - a_i) f'(a_i) \leq u(x) - u(a_i) \leq (x - a_i) f'(b_i)$$

whence

$$\mathrm{H} \equiv \frac{f(a_i) + f'(a_i)(x - a_i) - f(x_0)}{x - x_0} \leq \frac{u(x) - u(x_0)}{x - x_0} \leq \mathrm{H} + \frac{(x - a_i)}{x - x_0}(f'(b_i) - f'(a_i)).$$

Since $f'(x)$ is continuous we have, as x, a_i, b_i tend to x_0,

$$f(a_i) + f'(a_i)(x - a_i) + \mathrm{o}(|x - a_i|) = f(x) \quad , \quad f'(b_i) - f'(a_i) = \mathrm{o}(b_i - a_i)$$

so that

$$\mathrm{H} = \frac{f(x) - f(x_0)}{x - x_0} + \mathrm{o}(1).$$

Accordingly if x tends to x_0 from the right, but $x \in \complement\mathrm{S}$, we have $\lim_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0} = f'(x_0)$, while the same relation is obvious if $x \in \mathrm{S}$. Lemma 1.3 follows.

§ 2. *Continuity of* $\frac{\mathrm{d}u}{\mathrm{d}x}$.

The results of § 1 guarantee the existence of $\frac{\mathrm{d}u}{\mathrm{d}x}(x)$ if x is restricted to the interval I minus the countable set of endpoints of component intervals of $\complement$S; at such points x_0 only $\mathrm{d}u/\mathrm{d}x^+$ and $\mathrm{d}u/\mathrm{d}x^-$ have been shown to exist, with one of these values equalling $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$ and the other being increasing (resp. decreasing) limit of $\frac{\mathrm{d}u}{\mathrm{d}x}(x)$ as $x \uparrow x_i$ (resp. $x \downarrow x_i$), $x \in \complement\mathrm{S}$. It remains to show that for such x_0 we have $\frac{\mathrm{d}u}{\mathrm{d}x^-}(x_0) = \frac{\mathrm{d}u}{\mathrm{d}x^+}(x_0) = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$. Suppose that this has been done; the continuity of $\frac{\mathrm{d}u}{\mathrm{d}x}(x)$ then follows for every $x \in \mathrm{I}$. For if $x \in \mathrm{S}$ and $x_i \to x$, $x_i \in \mathrm{S}$, then $\frac{\mathrm{d}u}{\mathrm{d}x}(x_i) = f'(x_i) \to f'(x)$, while if $x_i \in \complement\mathrm{S}$, $x_i \to x$, $a_i < x_i < b_i$, with (a_i, b_i) a component of $\complement$S, then $a_i \to x$, $b_i \to x$, and $f'(a_i) < \frac{\mathrm{d}u}{\mathrm{d}x}(x_i) < f'(b_i)$ and $f'(a_i) \to f'(x)$ and $f'(b_i) \to f'(x)$; on the other hand the continuity of $\frac{\mathrm{d}u}{\mathrm{d}x}(x)$ for x in the open set $\complement$S is trivial. With the proof of the following Lemma we therefore shall have proved Theorem 1.1.

LEMMA 2.1. *Let* $x_0 \in S$ *be right (resp. left) end point of a component of* $\complement S$. *Then* $\frac{du}{dx^-}(x_0) = f'(x_0)$ $\left(\text{resp. } \frac{du}{dx^+}(x_0) = f'(x_0)\right)$.

Proof. It suffices to prove Lemma 2.1 for x_0 being a right endpoint, and to assume for simplicity of notation that $x_0 = 0$. If $\rho > 0$ is small enough the interval $(-\rho, 0) \subset \complement S$. We extend $u(x)$ into a complex $z = x + ix^*$ plane by setting

$$u(z) = \int_S -\log|z-y|\, d\mu(y)$$

so that $u(z)$ is harmonic outside the set S of the real axis. We map $|z| < \rho$, $x^* > 0$ conformally onto the first quadrant of a $\xi + i\eta$–plane in such a way that $0 \to 0$, $\rho \to 1$, $-\rho \to i\infty$. A reflection in the interval $(-\rho, 0)$ corresponds to a reflection of the conformal map in the imaginary or η–axis of the $\xi + i\eta$–plane, yielding the conformal map of the circle $|z| < \rho$ slitted along the positive real axis into the upper half $\xi + i\eta$–plane. Put

$$u(z) = U(\xi + i\eta).$$

U is harmonic in ξ and η for $\eta > 0$ and assumes continuous (and bounded) boundary values; moreover the relation $u(z) = u(\bar{z})$ implies that $U(\xi) = U(-\xi)$ holds on $\eta = 0$.

For the inverse map M we have with a constant $k > 0$

$$z = x + ix^* = (\xi + i\eta)^2 (k + o(1)) \quad \text{as} \quad \xi + i\eta \to 0 \tag{2.1}$$

because another reflection in the positive x–axis (resp. the interval $-1 < \xi < 1$ of $\eta = 0$) shows M to be a conformal map of the $\xi + i\eta$–plane, slitted along the real axis from $-\infty$ to -1 and from 1 to ∞, onto the doubly covered circle $|z| < \rho$ with the origins corresponding.

Put $\Theta(\xi + i\eta; t) =$ angle from the negative η–axis to the vector $t - (\xi + i\eta)$, where t is real, $\eta > 0$. Then Poisson's formula gives

$$U(\xi + i\eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} U(t)\, d_t \Theta(\xi + i\eta; t).$$

The proven existence of $\frac{du}{dx^+}(0) = c = f'(0)$ implies $u(x) - u(0) = cx + o(x)$ as $x \downarrow 0$ which is translated by (2.1) into

$$U(\xi) - U(0) = \xi^2 (ck + o(1)) \quad \text{as} \quad \xi \to 0. \tag{2.2}$$

We claim that $\lim\limits_{\eta \downarrow 0} \frac{U(i\eta) - U(0)}{\eta} = 0$. Indeed this is equivalent

$$\lim_{\eta \downarrow 0} \frac{u(-\eta^2 k) - u(0)}{\eta k} = \lim_{\eta \downarrow 0} \eta \lim_{\eta \downarrow 0} \frac{u(-\eta^2 k) - u(0)}{\eta^2 k}$$

$$= \lim_{\eta \downarrow 0} \eta \lim_{x \uparrow 0} \frac{u(x) - u(0)}{-x} = 0$$

since we know $\frac{du}{dx^-}(0)$ to exist.

This relation $\frac{\partial U}{\partial \eta^+}(0) = 0$ is expressed, with $\tilde{U}(t) = U(t) - U(0)$, as

$$0 = \lim_{\eta \downarrow 0} \frac{\partial U}{\partial \eta}(i\eta) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{U}(t) \frac{t^2 - \eta^2}{(t^2 + \eta^2)^2} \, dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{U}(t) \frac{dt}{t^2},$$

account being taken of (2.2).

Now form

$$\frac{1}{\eta}\left(\frac{\partial U}{\partial \eta}(i\eta) - \frac{\partial U}{\partial \eta^+}(0)\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{U}(t) \frac{t^2(t^2 - \eta^2) - (t^2 + \eta^2)^2}{\eta(t^2 + \eta^2)^2 t^2} \, dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} -\tilde{U}(t) \frac{3t^2\eta + \eta^3}{(t^2 + \eta^2)^2 t^2} \, dt.$$

Note that the limit of this integral as $\eta \downarrow 0$ depends only on the values of $\tilde{U}(t)$ in an arbitrarily small neighborhood N of $t = 0$ which we choose so that in N

$$|\tilde{U}(t) - ckt^2| \leq \varepsilon t^2$$

where $\varepsilon > 0$ is given in advance. An easy evaluation yields

$$\lim_{\eta \downarrow 0} \frac{1}{\pi} \int_N \frac{-ckt^2(3\eta t^2 + \eta^3)}{(t^2 + \eta^2)^2 t^2} \, dt = -\frac{ck}{\pi} \int_{-\infty}^{\infty} \frac{3\tau^2 + 1}{(\tau^2 + 1)^2} \, d\tau = -2ck,$$

while

$$\overline{\lim} \frac{1}{\pi} \left| \int_N o(1)\, t^2 \frac{(3t^2\eta + \eta^3)}{(t^2 + \eta^2)^2 t^2} \, dt \right| \leq 2\varepsilon.$$

Hence $\lim_{\eta \downarrow 0} \frac{1}{\eta}\left(\frac{\partial U}{\partial \eta}(i\eta) - \frac{\partial U}{\partial \eta^+}(0)\right) = \lim_{\eta \downarrow 0} \frac{1}{\eta} \frac{\partial U}{\partial \eta}(i\eta) = -2ck = -2kf'(0)$.

But $\lim_{\eta \downarrow 0} \frac{1}{-2k\eta} \frac{\partial U}{\partial \eta}(i\eta) = \lim_{x \uparrow 0} \frac{du}{dx}(x) = \frac{du}{dx}(0)$.

Lemma 2.1 is proved and Theorem 1.1 is established.

§ 3. The proof of Lemma 2.1 given in § 2 necessitated the excursion into $\mathbf{R}^2$. The following proof permits us to remain in the $\mathbf{R}^1$ containing the mass of the potential.

For this purpose we utilize the function

$$\lambda(t) = \int_0^1 \frac{-\log|t - x|}{\sqrt{x(1-x)}} \, dx$$

which is continuous for $-\infty < t < \infty$, and has the property $\lambda(t+1) \leq \lambda(t)$ if $t \geq 0$, and $\lambda(t) = \lambda(1-t)$ for all real t.

Let now μ be a non-negative mass restricted to a bounded portion of the positive x–axis and such that its potential $u(x) = \int_0^\infty -\log|y-x|\,d\mu(y)$ has the property of possessing the two one-sided derivatives $\frac{du}{dx^+}(0) = c_+$ and $\frac{du}{dx^-}(0) = \int_0^\infty \frac{d\mu(y)}{y} = c_-$. We need prove $c_+ = c_-$. Evidently

$$\frac{\pi}{2} c_+ = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \frac{dx}{\sqrt{x(t-x)}} \int_0^\infty d\mu(y)\,(\log y - \log|y-x|) \tag{3.1}$$

$$-\frac{\pi}{2} c_- = \lim_{t\uparrow 0} \frac{1}{-t} \int_0^{-t} \frac{dx}{\sqrt{x(-t-x)}} \int_0^\infty d\mu(y)\,(\log y - \log|y-x|) \tag{3.2}$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \frac{dx}{\sqrt{x(t-x)}} \int_0^\infty d\mu(y)\,(\log y - \log|y+x|)$$

the factor $\pi/2$ being the value of $\int_0^1 \frac{x\,dx}{\sqrt{x(1-x)}}$. Subtracting (3.2) from (3.1) we obtain

$$\frac{\pi}{2}(c_+ + c_-) = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \frac{dx}{\sqrt{x(t-x)}} \int_0^\infty (-\log|y-x| + \log|y+x|)\,d\mu(y)$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \frac{d\xi}{\sqrt{\xi(1-\xi)}} \int_0^\infty (-\log|y-t\xi| + \log|y+t\xi|)\,d\mu(y)$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \int_0^\infty \left(\lambda\left(\frac{y}{t}\right) - \lambda\left(\frac{-y}{t}\right)\right) d\mu(y)$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \int_0^\infty \left(\lambda\left(\frac{y}{t}\right) - \lambda\left(1+\frac{y}{t}\right)\right) d\mu(y).$$

Now for each $y > 0$, $\lambda\left(\frac{y}{t}\right) - \lambda\left(1+\frac{y}{t}\right) \geq 0$ and

$$\lim_{t\downarrow 0} \frac{1}{t}\left(\lambda\left(\frac{y}{t}\right) - \lambda\left(1+\frac{y}{t}\right)\right) = \lim_{s\to\infty} \frac{s}{y}\,(\lambda(s) - \lambda(1+s)).$$

For $s \geq 0$, the expression $s(\lambda(s) - \lambda(1+s))$ is bounded since $\lambda(s)$ is continuous and the limit as $s \to \infty$ exists:

$$\lim_{s\to\infty} s(\lambda(s) - \lambda(1+s)) = \lim_{s\to\infty} \int_0^1 s \log \frac{1+s-x}{s-x} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi .$$

The dominated convergence theorem now yields

$$\frac{\pi}{2}(c_+ + c_-) = \pi \int_0^\infty \frac{d\mu(y)}{y} = \pi c_-$$

and thus

$$c_+ = c_- .$$

§ 4. We return to the problem mentioned in the introduction. With Ω a bounded smooth domain of $\mathbf{R}^2$, let $\psi(x)$ be a C^1 function defined on a line $\mathbf{R}^1 \cap \Omega$ and negative on $\mathbf{R}^1 \cap \partial\Omega$. Let $v(x)$ be the smallest superharmonic, continuous on $\Omega \cup \partial\Omega$, ≥ 0 on $\partial\Omega$ and $\geq \psi(x)$ on $\mathbf{R}^1 \cap \Omega$. With the aid of Green's function of Ω we can represent $v(x)$ as

$$v(x) = \int_{\mathbf{R}^1 \cap \Omega} G(x, y)\, d\mu(y)$$

where μ is a non-negative mass whose support S is a compact part of $\mathbf{R}^1 \cap \Omega$. Now $G(x, y) = -\log|x - y| + h(x, y)$ where $h(x, y)$ is continuous in x and y and harmonic in each variable ranging over Ω. If x and y are restricted to compact portions of Ω, the derivatives of all orders of h are bounded. Therefore $\int_S h(x, y)\, d\mu(y)$ is certainly a C^2 function in a neighborhood of S. The relation $v(x) \geq \psi(x)$ thus becomes

$$u(x) \equiv v(x) - \int_S h(x, y)\, d\mu(y) = \int_S -\log|x - y|\, d\mu(y) \geq$$

$$\geq \psi(x) - \int_S h(x, y)\, d\mu(y) = f(x)$$

and we know that the mass is carried by a point set for which $v(x) = \psi(x)$ or $u(x) = f(x)$. Since $\psi \in C^1(\mathbf{R}^1 \cap \Omega)$, it follows that $f(x) \in C^1$ in a (one-dimensional) interval neighborhood I of S. From Theorem 1.1 we infer that $u(x)$, and hence $v(x)$, are in $C^1(\mathbf{R}^1 \cap \Omega)$.

Suppose $\frac{d\psi}{dx}(x)$ is of bounded variation on $\mathbf{R}^1 \cap \Omega$. It then becomes possible to bound the total variation of $\frac{dv}{dx}(x)$ on I. In fact we know that

$\frac{du(x)}{dx}$ is monotone in $\complement S$ while coinciding with $\frac{df}{dx}(x)$ in S. Thus the total variation of $\frac{du}{dx}$ between the extremes of S is $\leq$ that of $\frac{df}{dx} \leq$ that of $\frac{d\psi}{dx}$ + that of $\int\limits_S \frac{\partial h}{\partial x}(x, y)\, d\mu(y)$.

Since S lies on $\{x : \psi(x) \geq 0\}$ and $\mu(S)$ can be estimated in terms of S, max ψ and Ω and $\mathbf{R}^1 \cap \Omega$, we see that the total variation of $\frac{dv}{dx}$ between the extremes of S is bounded, and hence easily also on I, if the closure of I is contained in Ω. The bond depends only on Ω, the sets $\{x : \psi(x) \geq 0\}$ and I, the maximum of ψ, and the total variation of $\frac{d\psi}{dx}$ on $\{x : \psi(x) \geq 0\}$.

THEOREM 4.1. *Let Ω be a bounded (smooth) domain of $\mathbf{R}^2$, and $\psi(x)$ a C^1 function of compact support defined on the intersection of a line (x–axis) with Ω. The smallest continuous super-harmonic $v(z)$ which $\geq \psi(x)$ on the x–axis and ≥ 0 on $\partial\Omega$, has a C^1 restriction to the x–axis. Moreover the total variation $\int \left| d\frac{dv}{dx}(x)\right|$ is bounded if $\int \left| d\frac{d\psi}{dx}\right|$ is bounded.*

If we apply to a circle of $\mathbf{R}^2$ a conformal transformation onto the plane slitted along the real axis from $-\infty$ to 1 and from 1 to ∞, Theorem 4.1 yields by an easy argument:

THEOREM 4.2. *Consider the class Γ of bounded harmonics h in the upper half plane Ω which are continuous on $\Omega \cup \partial\Omega$ and vanish on the segments $(-\infty, -1]$ and $[1, \infty)$ and are $\geq \psi(x)$ on $(-1, 1)$ where $\psi(x)$ is a given $C^1(-1, 1)$ function of compact support; and such that the conjugate harmonic of h increases monotonely on $(-1, 1)$. There exists a smallest $v(z)$ of Γ, and $v(x)$ is $C^1(-1, 1)$. If $\int_{-1}^{1} \left| d\frac{d\psi}{dx}\right| < \infty$, then $\int_{-1}^{1} \left| d\frac{dv}{dx}\right|$ is bounded in terms of the maximum of ψ, the support of ψ, and $\int_{-1}^{1} \left| d\frac{d\psi}{dx}\right|$.*

In Theorem 4.1 the line carrying the mass may be replaced by any Jordan arc of bounded curvature. The proof will appear elsewhere.

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