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On the optimization in distributed parameter control systems

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Teoria dei controlli. — *On the optimization in distributed parameter control systems* (*). Nota di MEHMET NAMIK OĞUZTÖRELI (**), presentata (***) dal Socio M. PICONE.

RIASSUNTO. — L'A. stabilisce in questo lavoro l'esistenza e l'unicità dei controlli ottimali per un sistema ordinario con parametri distribuiti spettante ad un funzionale quadratico – il costo prescelto.

In this paper we study the existence and uniqueness of optimal controls for a one-dimensional linear distributed parameter control system with a quadratic cost functional. Admissible controls satisfy certain boundary and integral constraints. The trigonometric method in [1]–[3] is used in the construction of the optimal control. It is shown that optimal controls satisfy an integro-partial differential equation of the second order.

1. INTRODUCTION.

Consider a one-dimensional distributed parameter control system defined for $0 \leq x \leq \pi$ and $0 \leq t \leq \pi$, where x and t are the spatial and time variables, respectively. Put

$$R = \{(t, x) \mid 0 \leq t \leq \pi, 0 \leq x \leq \pi\}. \tag{1.1}$$

The system is described by an *input-output* relationship of the form

$$u(t, x) = \Phi(t, x) + \iint_R K(t, x; \tau, \xi)\, v(\tau, \xi)\, d\xi\, d\tau \tag{1.2}$$

for $(t, x) \in R$, where $v(t, x)$ and $u(t, x)$ represent the input (*control*) and the output variables at (t, x), and $\Phi(t, x)$ and $K(t, x; \tau, \xi)$ are given functions which are continuous for $(t, x), (\tau, \xi) \in R$. A control $v(t, x)$ is said to be *admissible* if it is continuous and twice differentiable with respect to x on R and if it satisfies the conditions

$$v(t, 0) = v(t, \pi) = 0 \qquad \text{for} \quad 0 \leq t \leq \pi, \tag{1.3}$$

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and

$$\iint_R \left[\frac{\partial v(t, x)}{\partial x}\right]^2 dx\, dt = 1 . \tag{1.4}$$

We denote by V the set of all admissible controls.

The performance of the system under a control $v(t, x)$ is measured by the *cost functional*

$$J(v) = \iint_R \{u^2(t, x) + v^2(t, x)\}\, dx\, dt, \tag{1.5}$$

where $u(t, x)$ is given by (1.2).

In this paper we deal with the following optimization problem:

Find an admissible control $v^0(t, x)$ *for which the cost functional* $J(v)$ *achieves its minimum on* V.

Such a control will be called *optimal.*

Clearly, the method of *Lagrange* multipliers can be employed in the optimization problem formulated above. Accordingly, consider the functional

$$F(v) = \iint_R \left\{ u^2(t, x) + v^2(t, x) - \mu \left[\frac{\partial v(t, x)}{\partial x}\right]^2 \right\} dx\, dt, \tag{1.6}$$

where μ is the *Lagrange* multiplier. The *Euler-Lagrange* equation associated with the functional $F(v)$ is the following integro-partial differential equation:

$$\mu \frac{\partial^2 v(t, x)}{\partial x^2} - v(t, x) = \Phi_1(t, x) + \iint_R K_1(t, x; \sigma, \eta)\, v(\sigma, \eta)\, d\eta\, d\sigma, \tag{1.7}$$

where

$$\Phi_1(t, x) = \iint_R K(\tau, \xi; t, x)\, \Phi(\tau, \xi)\, d\xi\, dt, \tag{1.8}$$

and

$$K_1(t, x; \sigma, \eta) = \iint_R K(t, x; \tau, \xi)\, K(\tau, \xi; \sigma, \eta)\, d\xi\, d\tau. \tag{1.9}$$

We shall see in the next section that $\mu > 0$. Thus, any optimal control satisfies Eq. (1.7) with a certain μ, subject to the conditions (1.3).

2. Construction of an Optimal Control.

Consider the *Hilbert* space $\mathfrak{V} \equiv L^2(R)$ of functions $v = v(t, x)$ which are square integrable on R. We have $V \subset \mathfrak{V}$, and the functional $J(v)$ is strongly continuous on V and $J(v) > 0$ for all $v \in V$.

Consider the functions

$$v_n(t,x) = \sum_{j,k=1}^{n} \alpha_{jk} \sin jt \sin kx , \qquad (n = 1, 2, 3, \cdots) \tag{2.1}$$

where the α_{jk}'s are constants. Clearly, any $v_n(t,x)$ satisfies the conditions (1.3). Hence $v_n(t,x)$ is admissible if it satisfies the condition (1.4), that is, if

$$\iint_R \left\{ \sum_{j,k=1}^{n} k\alpha_{jk} \sin jt \cos kx \right\}^2 dx\, dt = \frac{\pi^2}{4} \sum_{j,k=1}^{n} k^2 \alpha_{jk}^2 = 1 . \tag{2.2}$$

Put $\alpha = (\alpha_{jk})$, $j, k = 1, \cdots, n$, and

$$f_n(\alpha) \equiv J(v_n) . \tag{2.3}$$

The function $f_n(\alpha)$ is a positive definite quadratic form in the variables α_{jk}. Since $f_n(\alpha)$ is continuous on the surface σ_{n^2} of the n^2–dimensional ellipsoid defined by (2.2) and σ_{n^2} is compact, $f_n(\alpha)$ achieves its minimum at some point $\alpha^0 \in \sigma_{n^2}$. Put

$$v_n^0(t,x) = \sum_{j,k=1}^{n} \alpha_{jk}^0 \sin jt \sin kx \tag{2.4}$$

and

$$\mu_n^0 = f_n(\alpha^0) = \min_{\alpha \in \sigma_{n^2}} f_n(\alpha) = J(v_n^0) . \tag{2.5}$$

We have $\mu_n^0 > 0$ and $\mu_{n+1}^0 \le \mu_n^0$ for any n. Hence the sequence $\{\mu_n^0\}$ is convergent:

$$\mu^0 = \lim_{n\to\infty} \mu_n^0 = \lim_{n\to\infty} J(v_n^0) . \tag{2.6}$$

Note that $\mu^0 > 0$. By virtue of the convergence of the sequence $\{\mu_n^0\}$ and of the condition (1.4), we see that the sequences $\{u_n^0(t,x)\}$, $\{v_n^0(t,x)\}$ and $\left\{\frac{\partial v_n^0(t,x)}{\partial x}\right\}$ are all *bounded*, and therefore *weakly compact* in V. Thus we can extract subsequences from these sequences, converging weakly to certain functions $u^0(t,x)$, $v^0(t,x)$ and $h(t,x)$, respectively, belonging to V:

$$u_n^0(t,x) \equiv \Phi(t,x) + \iint_R K(t,x;\tau,\xi)\, v_n^0(\tau,\xi)\, d\xi\, d\tau \xrightarrow{w} u^0(t,x)$$

$$\equiv \Phi(t,x) + \iint_R K(t,x;\tau,\xi)\, v^0(\tau,\xi)\, d\xi\, d\tau ,$$

$$v_n^0(t,x) \xrightarrow{w} v^0(t,x) \quad , \quad \frac{\partial v_n^0(t,x)}{\partial x} \xrightarrow{w} h(t,x) , \tag{2.7}$$

with

$$\iint_R \{[u^0(t,x)]^2 + [v^0(t,x)]^2\}\,dx\,dt = \mu^0 \quad , \quad \iint_R h^2(t,x)\,dx\,dt = 1. \tag{2.8}$$

In the following, we show that the function $v^0(t,x)$ is the required optimal control.

First of all, $v^0(t,x)$ satisfies the conditions (1.3). We now show that $v^0(t,x)$ is twice differentiable with respect to x in R, and

$$h(t,x) = \frac{\partial v^0(t,x)}{\partial x}, \tag{2.9}$$

almost everywhere in R. Indeed, we know that the positive definite form $f_n(\alpha)$ assumes its minimum μ_n^0, subject to the constraint (1.4), at the point $\alpha^0 \in \sigma_{n^2}$. By the method of *Lagrange* multipliers we have

$$\frac{\partial}{\partial\alpha_{jk}}\left\{ f_n(\alpha) - \mu_n^0 \iint_R \left[\sum_{j,k=1}^{n} k\alpha_{jk} \sin jt \sin kx\right]^2 \right\} dx\,dt = 0 \tag{2.10}$$

for $\alpha = \alpha^0$, which yield the equations

$$\iint_R \left\{ u_n^0(t,x)\frac{\partial u_n^0(t,x)}{\partial\alpha_{jk}} + v_n^0(t,x)\sin jt \sin kx - \mu_n^0 \frac{\partial v_n^0(t,x)}{\partial x} k \sin jt \cos kx \right\} dx\,dt = 0 \tag{2.11}$$

for $j, k = 1, \cdots, n$. Multiplying each of the equations (2.11) by an arbitrary constant $y_{jk}^{(n)}$ and summing over j and k from 1 to n, and manipulating on the first term, we obtain

$$\iint_R \left\{ \left[\Phi_1(t,x) + \iint_R K_1(t,x;\sigma,\eta)\, v_n^0(\sigma,\eta)\,d\sigma\,d\eta + v_n^0(t,x)\right] Q_n(t,x) - \mu_n^0 \frac{\partial v_n^0(t,x)}{\partial x}\frac{\partial Q_n(t,x)}{\partial x} \right\} dx\,dt = 0, \tag{2.12}$$

where

$$Q_n(t,x) = \sum_{j,k=1}^{n} \gamma_{jk}^{(n)} \sin jt \sin kx. \tag{2.13}$$

Integrating by parts the last term in (2.12), and taking into account the conditions (1.3), we find

$$\iint_R \left\{ \left[\Phi_1(t,x) + \iint_R K_1(t,x;\sigma,\eta)\, v_n^0(\sigma,\eta)\,d\eta\,d\sigma + v_n^0(t,x)\right] Q_n(t,x) - \mu_n^0 v_n^0(t,x) \frac{\partial^2 Q_n(t,x)}{\partial x^2} \right\} dx\,dt = 0. \tag{2.14}$$

We now select arbitrarily a twice continuously differentiable function $Q(t, x)$ such that

(2.15) $$Q(t, 0) = Q(t, \pi) = 0 \quad \text{for} \quad 0 \leq t \leq \pi .$$

Clearly, we can choose the coefficients $\gamma_{jk}^{(n)}$ in such a manner that

(2.16) $$\lim_{n \to \infty} \iint_R |Q_n^{(i)}(t, x) - Q^{(i)}(t, x)|^2 \, dx \, dt = 0 \qquad (i = 0, 1, 2)$$

where $Q_n^{(i)}(t, x)$ and $Q^{(i)}(t, x)$ denote the partial derivatives with respect to x of $Q_n(t, x)$ and $Q(t, x)$ of order i, respectively. Then, passing to the limit as $n \to \infty$ in (2.12) and (2.14), and taking into account the relations (2.6) and (2.7), we find

(2.17) $$\iint_R \left\{ w(t, x) Q(t, x) - \mu^0 h(t, x) \frac{\partial Q(t, x)}{\partial x} \right\} dx \, dt = 0$$

and

(2.18) $$\iint_R \left\{ w(t, x) Q(t, x) - \mu^0 v^0(t, x) \frac{\partial^2 Q(t, x)}{\partial x^2} \right\} dx \, dt = 0,$$

where

(2.19) $$w(t, x) = \Phi_1(t, x) + v_0(t, x) + \iint_R K_1(t, x; \sigma, \eta) \, v^0(\sigma, \eta) \, d\eta \, d\sigma .$$

Clearly $w(t, x) \in L^2(R)$. Consider the function

(2.20) $$W(t, x) = -\int_0^\pi G(x, \xi) \, w(t, \xi) \, d\xi ,$$

where

$$G(x, \xi) = \begin{cases} \dfrac{\xi(\pi - x)}{\pi} & \text{for} \quad \xi \leq x , \\[2ex] \dfrac{x(\pi - \xi)}{\pi} & \text{for} \quad x \leq \xi . \end{cases}$$

We then have

(2.22) $$\begin{cases} \dfrac{\partial^2 W(t, x)}{\partial x^2} = w(t, x) & \text{for} \quad (t, x) \in R , \\[1ex] W(t, 0) = W(t, \pi) = 0 & \text{for} \quad 0 \leq t \leq \pi . \end{cases}$$

By virtue of Eq. (2.15) and (2.22), an integration by parts in the first term of Eq. (2.18) leads us to the following equality:

(2.23) $$\iint_R [W(t, x) - \mu^0 v^0(t, x)] \frac{\partial^2 Q(t, x)}{\partial x^2} \, dx \, dt = 0 .$$

Note that the function

$$Q(t,x) = -\int_0^{\pi} G(x,\xi)\,[W(t,\xi) - \mu^0 v^0(t,\xi)]\,d\xi \tag{2.24}$$

is in $L^2(R)$ and twice differentiable with respect to x on R. Further

$$\frac{\partial^2 Q(t,x)}{\partial x^2} = W(t,x) - \mu^0 v^0(t,x) \quad , \quad Q(t,0) = Q(t,\pi) = 0 \tag{2.25}$$

almost everywhere in R. Thus, combining Eqs. (2.23)–(2.25), we obtain

$$\iint_R [W(t,x) - \mu^0 v^0(t,x)]^2\,dx\,dt = 0. \tag{2.26}$$

Therefore

$$\mu^0 v^0(t,x) = W(t,x), \tag{2.27}$$

almost everywhere in R; and, since the function $W(t,x)$ satisfies Eq. (2.23), we have

$$\mu^0 \frac{\partial^2 v^0(t,x)}{\partial x^2} - v^0(t,x) = \Phi_1(t,x) + \iint_R K_1(t,x;\sigma,\eta)\,v^0(\sigma,\eta)\,d\eta\,d\sigma, \tag{2.28}$$

i.e., the function $v^0(t,x)$ is twice differentiable in R and it satisfies the *Euler-Lagrange* equation (1.7).

We now establish Eq. (2.9). To this end, let us integrate by parts Eq. (2.18). Then, on account of the conditions (1.3) and (2.15), we obtain

$$\iint_R \left\{ w(t,x)\,Q(t,x) - \mu^0 \frac{\partial v^0(t,x)}{\partial x} \frac{\partial Q(t,x)}{\partial x} \right\} dx\,dt = 0, \tag{2.29}$$

or, subtracting (2.17) form (2.29),

$$\iint_R \left[\frac{\partial v^0(t,x)}{\partial x} - h(t,x) \right] \frac{\partial Q(t,x)}{\partial x}\,dx\,dt = 0. \tag{2.30}$$

Since $Q(t,x)$ satisfies Eqs. (2.15) and is continuously differentiable in x, but otherwise arbitrary, we can easily show that Eq. (2.9) is valid almost everywhere in R.

Thus the control $v^0(t,x)$ is admissible and, by virtue of Eqs. (2.8) and the definition of μ^0, it is optimal, as asserted.

3. Uniqueness of the Optimal Controls.

In the previous section we constructed the optimal control $v^0(t,x)$. Suppose that we have another optimal solution, say $v^1(t,x)$. Consider the function $v(t,x) = v^0(t,x) - v^1(t,x)$. Since both $v^0(t,x)$ and $v^1(t,x)$

satisfy Eq. (1.7) with $\mu = \mu^0 > 0$, we have

$$\mu^0 \frac{\partial^2 v(t,x)}{\partial x^2} = v(t,x) + \iint_R K_1(t,x;\sigma,\eta)\, v(\sigma,\eta)\, d\eta\, d\sigma \tag{3.1}$$

for $(t,x) \in R$, and $v(t,0) = v(t,\pi) = 0$ for $0 \leq t \leq \pi$. According to Eqs. (2.20)–(2.22), the solution of Eq. (3.1) subject of the boundary conditions $v(t,0) = v(t,\pi) = 0$ for $0 \leq t \leq \pi$, satisfies the integral equation

$$\mu^0 v(t,x) = -\int_0^\pi G(x,\xi)\left[v(t,\xi) + \iint_R K_1(t,\xi;\sigma,\eta)\, v(\sigma,\eta)\, d\eta\, d\sigma\right] d\xi. \tag{3.2}$$

Thus the uniqueness of the optimal control $v^0(t,x)$ is equivalent of the uniqueness of the solution $v(t,x) \equiv 0$ of Eq. (3.2).

For further discussion on this subject we refer to [5].

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