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Asymptotic properties of solutions of a certain n—th order vector differential equation

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Equazioni differenziali. — *Asymptotic properties of solutions of a certain n–th order vector differential equation* (*). Nota di STANISŁAW SĘDZIWY, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota dà condizioni sufficienti per la limitatezza delle soluzioni di un sistema di equazioni differenziali ordinarie (1) nel caso in cui $f(x)$ è limitata.

1. Consider the system of n–th order differential equations

$$x^{(n)} + A_1 x^{(n-1)} + \cdots + A_{n-1} x' + f(x) = p(t) \ ('= d/dt), \tag{1}$$

where $x \in R^m$ (R^m denotes the m–dimensional real Euclidean space with the norm $|x|$), $m \times m$ matrices A_i are constant and functions $f: R^m \to R^m$, $p: R \to R^m$ are continuous.

The solutions of (1) are said to be globally bounded if there exists a $\delta > 0$, such that any solution $x = x(t)$ of (1) satisfies $\sum_{i=0}^{n-1} |x^{(i)}(t)| < \delta$ for $t \geq T$, where T depends only on $x(t)$.

This Note presents the sufficient conditions for the global boundedness of solutions of (1) in the case when $f(x)$ is bounded. This problem has been considered by [2], [3], [4], [5]. This Note generalizes the results of [4] (Th. 1, 2) to the vector differential equation (1).

We will use the following notations: capital Latin letters denote matrices, small Latin letters denote column-vectors, small Greek letters denote scalars. Exceptions: letters t, W, V and the notation of indices.

A^* denotes the matrix transpose of A. (a, b) is a scalar product of vectors a, b. A symmetric matrix A is said to be positive definite ($A > 0$) if $(x, Ax) > 0$ for $|x| \neq 0$. I_m is the $m \times m$ unit matrix.

2. THEOREM 1. *Let the polynomial* $\varphi(\lambda) = \mathrm{Det}(\lambda^{n-1} I_m + \lambda^{n-2} A_1 + \cdots + A_{n-1})$ *have the roots with negative real parts. Let* A_{n-1} *be symmetric and positive definite. Let* $f(x)$ *satisfy*

$$|f(x)| \leq \mu_1 \quad \text{for all } x, \tag{2}$$

$$\lim_{|x| \to \infty} (f(x) - q, x) = \infty \quad \text{for all } q \text{ such that } |q| \leq \mu_2. \tag{3}$$

If $|p(t)| \leq \mu_2$ for $t \geq 0$, then solutions of (1) are globally bounded.

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Proof. We will prove the boundedness of solutions of the equivalent system

$$(4)\qquad y' = Ay + B\,(f(s) - p(t)), \quad s' = Cy,$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, y_1, \cdots, y_{n-1} \in R^m, \quad s = x \text{ and } B, C, A \text{ are}$$

$m(n-1)\times m$, $m\times(n-1)m$, $m(n-1)\times m(n-1)$ matrices of the form

$$B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -I_m \end{pmatrix}, \quad C^* = \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0, I_m, 0 & ,\cdots, & 0 \\ 0, 0, I_m, 0, & \cdots, & 0 \\ \cdots & \cdots & \cdots \\ 0 & ,\cdots, & 0, I_m \\ -A_{n-1} & ,\cdots, & -A_1 \end{pmatrix}.$$

The proof consists of a construction in the (y, s)–space of a bounded set Δ with the property that every solution $y = y(t)$, $s = s(t)$ of (4) starting at $t = t_0$ exists for $t \geq t_0$ and $(y(t), s(t)) \in \Delta$ for large t.

Put $V(y) = (y, Ly)$, $W(y, s) = (Ry, s) + (Ps, s)$, where $L^* = L > 0$, $P^* = P > 0$ and L, P, R satisfy

$$(5)\qquad A^*L + LA = -Q\ (Q^* = Q > 0),$$

$$(6)\qquad RA + 2\,PC = 0, \ RB = -I_m.$$

Since $\mathrm{Det}\,(A - \lambda I_{m(n-1)}) = \varphi(\lambda)$, A has eigenvalues with negative real parts. Hence for $Q > 0$, arbitrarily chosen there exists a symmetric, positive definite matrix L satisfying (5) (see [1]).

(6) has a solution $P = \frac{1}{2}(CA^{-1}B)^{-1}$, $R = -2\,PCA^{-1}$ An easy calculation shows that $CA^{-1}B = A_{n-1}^{-1}$, hence $P^* = P > 0$.

Let $\Delta(\alpha, \beta) = \{(y, s) : V(y) \leq \alpha, W(y, s) \leq \beta\}$. Since

$$\lim_{|y| \to \infty} V(y) = \infty,$$

$$\lim_{|s| \to \infty} W(y, s) = \infty \text{ uniformly for } y \text{ in an arbitrary compact set,}$$

$\Delta(\alpha, \beta)$ is compact for $\alpha, \beta > 0$.

For fixed $\mu > 0$ choose α_0, β_0 such that

$$(7)\qquad -(Qy, y) + 2\,(y, LB\,(f(s) - p(t))) \leq -\mu \text{ for } V(y) = \alpha_0,\ t, s \text{ arbitrary,}$$

$$(8)\qquad (Ry, Cy) - (f(s) - p(t), s) \leq -\mu \text{ for } V(y) = \alpha_0,\ W(y, s) = \beta_0,\ t \text{ arbitrary.}$$

The possibility of the choice of α_0, β_0 follows from (2), (3) and the boundedness of the form (Ry, Cy) for y satisfying $V(y) = \alpha_0$.

Let $y = y(t)$, $s = s(t)$ be the solution of (4), satisfying the initial conditions $y(t_0) = y_0$, $s(t_0) = s_0$. We will prove that $(y(t), s(t)) \in \Delta(\alpha_0, \beta_0)$ for large t. To this end, chose $\alpha_1 \geq \alpha_0$, $\beta_1 \geq \beta_0$ such that $(y_0, s_0) \in \Delta(\alpha_1, \beta_1)$ and

(9) $(Ry, Cy) - (f(s) - p(t), s) \leq -\mu$ for $V(y) = \alpha_1$, $W(y, s) = \beta_1$, t arbitrary.

Let $V(t) = V(y(t))$, $W(t) = W(y(t), s(t))$. By (5), (6),

$$V'(t) = -(Qy(t), y(t)) + 2(y(t), LB(f(s(t)) - p(t))),$$

$$W'(t) = (Ry(t), Cy(t)) - (f(s(t)) - p(t), s(t)).$$

Hence $\alpha_1 \geq \alpha_0$, (7), (9) imply that $(y(t)) \in \Delta(\alpha_1, \beta_1)$ for $t \geq t_0$. Observe that (7) and (8) imply that if $(y_0, s_0) \in \Delta(\alpha_0, \beta_0)$, then $(y(t), s(t))$ remains in $\Delta(\alpha_0, \beta_0)$ for $t \geq t_0$.

By (7), $V' \leq -\mu < 0$ for $V(t) > \alpha_0$. So $V(t) \leq \alpha_0$ for $t \geq T_1 = t_0 + (\alpha_1 - \alpha_0)/\mu$. It is clear that (9) holds for $V(y) \leq \alpha_0$, $\beta_0 < W(y, s) \leq \beta_1$. Thus $W'(t) \leq -\mu < 0$ for $t \geq T_1$, which implies that there is a $T_2 \geq T_1$ such that $W(t) \leq \beta_0$ for $t \geq T_2$.

Since $\Delta(\alpha_0, \beta_0)$ is bounded, $(y(t), s(t))$ exists for $t \geq t_0$ and satisfies $|y(t)| + |s(t)| < \delta$ for suitably chosen δ. This ends the proof.

3. If $p(t)$ and $\int_0^t p(\tau)\,d\tau$ are bounded, then Theorem 1 remains valid under the slightly weaker assumptions on f. Namely we have the following.

THEOREM 2. *Let assumptions of Theorem 1 hold with* (3) *replaced by*

(10) $$\lim_{|x| \to \infty} (f(x), x) = \infty.$$

If $|p(t)| \leq \mu_2$, $\left|\int_0^t p(\tau)\,d\tau\right| \leq \mu_3$ for $t \geq 0$, then solutions of (1) are globally bounded.

Proof. Let $V(y) = (y, Ly)$, $W_1(y, s, t) = \left(Ry - \int_0^t p(\tau)\,d\tau, s\right) + (Ps, s)$, where L, P, R satisfy (5), (6).

By (10) and the boundedness of $\int_0^t p(\tau)\,d\tau$,

$$\lim_{|s| \to \infty} W_1(y, s, t) = \infty,$$

$$\lim_{|s| \to \infty} \left[\left(Ry - \int_0^t p(\tau)\,d\tau, s\right) - (f(s), s)\right] = \infty$$

uniformly for $(y, t) \in \Omega \times [0, \infty)$, where Ω is an arbitrary compact set. Thus proof reduces to repeating the reasoning used in the proof of Theorem 1, with $W(y, s)$ replaced by $W_1(y, s, t)$ and is left to the reader.

4. THEOREM 3. *Assume the conditions of Theorem 1 or 2.*

If $p(t)$ *is periodic with period* ω *and, in the case of Theorem 2,* $\int_0^\omega p(t)\,dt = 0$, *then* (1) *has at least one periodic solution with period* ω.

Proof. Observe that $\Delta(\alpha, \beta) = \Delta_1(\alpha) \cap \Delta_2(\beta)$ or $\Delta(\alpha, \beta) = \Delta_1(\alpha) \cap \Delta_3(\beta)$, where

$$\Delta_1(\alpha) = \{(y, s) : V(y) \leq \alpha, \ s \text{ arbitrary}\},$$
$$\Delta_2(\beta) = \{(y, s) : W(y, s) \leq \beta\},$$
$$\Delta_3(\beta) = \{(y, s) : W_1(y, s, t) \leq \beta, \ t \text{ arbitrary}\}.$$

$(0, 0)$ is the interior point of Δ_i $(i = 1, 2, 3)$ and the Δ_i have the property that any half-line emanating from the origin of coordinates intersects the boundary of Δ_i at most at one point. Hence $\Delta(\alpha, \beta)$ is homeomorphic to the ball $(y, y) + (s, s) \leq 1$ and the assertion of Theorem 3 follows from Theorem 1 (Theorem 2) and Brouwer Fixed Point Theorem.

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