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Fixed points for densifying mappings

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Analisi matematica. — *Fixed points for densifying mappings* (*). Nota di MASSIMO FURI e ALFONSO VIGNOLI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Usando il numero α di Kuratowski si provano due teoremi di punto fisso per trasformazioni continue di uno spazio metrico completo in sé.

1. Let $T : X \to X$ be a continuous mapping of a complete metric space (c.m.s.) (X, d) into itself. In this paper we will give an existence theorem for fixed points of T.

To our purposes the following two definitions will be useful.

Definition 1. (C. Kuratowski [1]). Let $A \subset X$ be a bounded set. By the real number $\alpha(A)$ we denote the infimum of all numbers $\varepsilon > 0$ such that A admits a finite covering consisting of subsets of diameter less than ε.

It is easy to see that:

a) $\alpha(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of the set $A \subset X$;
b) $\alpha(A) = 0$ iff A is precompact;
c) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
d) $\alpha(B(A, \varepsilon)) \leq \alpha(A) + 2\varepsilon$, where $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$;
e) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$;
f) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, where A, B are subsets of a linear metric space.

Definition 2. (See [2]). A continuous mapping $T : D \to X$ defined on a subset D of a metric space X is called *densifying* if for every bounded set $A \subset D$, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Obviously, contractive mappings and completely continuous mappings are densifying.

2. The main result of the present paper is given by the following

THEOREM 1. *Let* (X, d) *be a metric space and let* $T : D \to X$ *be a densifying mapping defined on a complete subset* $D \subset X$. *Then any bounded sequence* $\{x_n\}$, *such that* $d(x_n, T(x_n)) \to 0$, *is compact and all the limit points of* $\{x_n\}$ *are fixed for* T.

Proof. Let $\{x_n\}$ be a bounded sequence such that $d(x_n, T(x_n)) \to 0$. Put $M = \{x_n : n = 1, 2, \cdots\}$; so that $T(M) = \{T(x_n) : n = 1, 2, \cdots\}$. Given any $\varepsilon > 0$, it follows that $B(T(M), \varepsilon)$ contains all but a finite number of

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elements of M, since $d(x_n, T(x_n)) \to 0$. Then $\alpha(M) \leq \alpha(B(T(M), \varepsilon)) \leq \leq \alpha(T(M)) + 2\varepsilon$; hence $\alpha(T(M)) \geq \alpha(M)$. This implies that $\alpha(M) = 0$, otherwise it would be $\alpha(T(M)) < \alpha(M)$. Therefore $\{x_n\}$ is compact. By the continuity of T all the limit points of $\{x_n\}$ are fixed for T.

Note that Theorem 1 can be formulated as follows:

Let $T : D \to X$ *be a densifying mapping defined on a complete subset* $D \subset X$ *of metric space* (X, d). *Then* T *has a fixed point if and only if there exists a bounded sequence* $\{x_n\}$ *such that* $d(x_n, T(x_n)) \to 0$.

COROLLARY 1. *Let* $T : X \to X$ *be a densifying mapping defined on a bounded (c.m.s.)* (X, d). *If* $\inf\{d(x, T(x)) : x \in X\} = 0$, *then* T *has at least a fixed point* $\xi \in X$.

Proof. It follows immediately from Theorem 1.

COROLLARY 2. *Let* $T : X \to X$ *be a completely continuous mapping defined on a bounded c.m.s.* (X, d). *If* $\inf\{d(x, T(x)) : x \in X\} = 0$, *then* T *has at least a fixed point* $\xi \in X$.

Proof. It follows immediately from Corollary 1.

COROLLARY 3. *Let* $T : D \to F$ *be a mapping defined on a closed subset* D *of a Fréchet space* F *such that* $T = G + H$, *where* $G : D \to F$ *is a completely continuous mapping and* $H : D \to F$ *is contractive (or, more generally, densifying). Then any bounded sequence* $\{x_n\}$ *such that* $d(x_n, T(x_n)) \to 0$ *is compact and all the limit points of* $\{x_n\}$ *are fixed for* T.

Proof. By Theorem 1 it is sufficient to prove that T is densifying. Let $A \subset F$ be a bounded set with $\alpha(A) > 0$. We have

$$\alpha(T(A)) \leq \alpha(G(A) + H(A)) \leq \alpha(G(A)) + \alpha(H(A)) = \alpha(H(A)) < \alpha(A).$$

3. Let $T : X \to X$ be a densifying mapping defined on a c.m.s. X. Let us consider in particular the sequence $\{T^n(x)\}$ of the iterates of x. In this section we impose some additional condition on the mapping T in order that for every x the sequence of iterates of x converges to the unique fixed point of T.

From Theorem 1 it follows.

THEOREM 2. *Let* $\varphi : R^+ \to R^+$ *be a right continuous non decreasing function defined on* $[0, +\infty]$ *such that* $\varphi(r) < r$ *for* $r > 0$. *Let* $T : X \to X$ *be a mapping defined on a* c.m.s. (X, d) *such that*

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \qquad \forall x, y \in X.$$

Then for any $x_0 \in X$ *the sequence* $\{x_n\}$ *of iterates starting from* x_0 $(x_1 = T(x_0), x_n = T(x_{n-1}))$ *converges to the unique fixed point of* T.

Proof. The mapping T is densifying. Indeed, let $A \subset X$ be a bounded set such that $0 < \alpha(A)$, and take $\varepsilon > \alpha(A)$; then there exists a finite covering $\{A_1, A_2, \cdots, A_n\}$ of subsets of A, such that $\delta(A_k) < \varepsilon$ $(k = 1, 2, \cdots, n)$.

Clearly

$$T(A) = \bigcup_{k=1}^{n} T(A_k) .$$

Let $1 \leq k \leq n$ be fixed and take $x, y \in A_k$. Then $d(T(x), T(y)) \leq \leq \varphi(d(x, y)) \leq \varphi(\varepsilon)$; hence $\delta(T(A_k)) \leq \varphi(\varepsilon)$ and therefore $\alpha(T(A)) \leq \varphi(\varepsilon)$. If $\varepsilon \downarrow \alpha(A)$, then, because of the right continuity of φ, we obtain $\alpha(T(A)) \leq \varphi(\alpha(A)) < \alpha(A)$, i.e. T is densifying.

Let $x_0 \in X$, and let $\{x_n\}$ be the sequence of iterates starting from x_0. Put $\beta_n = d(x_n, T(x_n))$. Let us prove that $\beta_n \to 0$. Clearly $\beta_n \leq \varphi(\beta_{n-1}) \leq \beta_{n-1}$, hence $\{\beta_n\}$ is a nonnegative nonincreasing sequence. Let $\beta_n \to \beta$; from the right continuity of φ it follows that $\beta \leq \varphi(\beta)$. This implies $\beta = 0$, otherwise it would be $\varphi(\beta) < \beta$.

Since T has at most one fixed point, by Theorem 1, it is sufficient to prove that the sequence $\{x_n\}$ is bounded. Let r be a positive real number. Since $\beta_n \to 0$, there exists a positive integer N such that $\beta_N < r - \varphi(r)$. Let us prove that the set $B(x_N, r)$ is invariant, i.e. $T(B(x_N, r)) \subset B(x_N, r)$. Indeed, take $y \in X$ such that $d(y, x_N) < r$, then

$$d(T(y), x_N) \leq d(T(y), T(x_N)) + d(T(x_N), x_N) \leq$$
$$\leq \varphi(d(y, x_N)) + \beta_N < \varphi(r) + (r - \varphi(r)) = r .$$

Clearly $x_n \in B(x_N, r)$ for $n \geq N$, i.e. $\{x_n\}$ is bounded.

From the above Theorem it follows immediately the Banach's contraction principle and, in the case that X is bounded, a fixed point theorem of F. E. Browder [3]. Another consequence of Theorem 2 is the following

COROLLARY 4. (Krasnosel'skij M. A. and Stecenko V. Ja. [4]). *Let* $T: X \to X$ *be a mapping defined on a* c.m.s. (X, d), *which satisfies the condition*

$$d(T(x), T(y)) \leq d(x, y) - \Delta(d(x, y)), \qquad \forall x, y \in X,$$

where Δ *is a continuous real function defined on* $[0, +\infty)$, *such that* $\Delta(r) > 0$ *for* $r > 0$. *Then for any* $x_0 \in X$, *the sequence* $\{x_n\}$ *of iterates starting from* x_0 *converges to the unique fixed point* $\xi \in X$.

Proof. Put $\varphi(r) = \max\{t - \Delta(t) : t \leq r\}$. It is easy to see that φ is continuous, nondecreasing and such that $\varphi(r) < r$ for $r > 0$. Obviously, $d(T(x), T(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$. Therefore from Theorem 2 the assertion follows.

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