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On special torsion-free groups

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Algebra. — *On special torsion-free groups.* Nota di ANTONIO MACHÍ [(*)] presentata [(**)] dal Socio B. SEGRE.

RIASSUNTO. — Si studia la classe dei gruppi privi di torsione che godono della seguente proprietà: dati comunque due elementi x e y esiste un intero positivo $n = n(x, y)$ tale che $x^n y = yx^n$. Si dà una condizione sufficiente perché tali gruppi siano abeliani. Si congettura, infine, che detti gruppi non possano essere semplici.

INTRODUCTION.

The purpose of this paper is to investigate the structure of torsion-free groups G with the following property:

$$G = \{x \in G \mid \forall g \in G \; \exists t = t(x, g) > 0 \text{ s.t. } xg^t = g^t x\}, \tag{1}$$

i.e., every element of G commutes with some positive power of every element. We will write G ∈ (C) if G is torsion-free and has property (1). Clearly, if H is a subgroup of G and G ∈ (C) then H ∈ (C).

It is interesting to compare the group situation with that of rings. There, by a theorem of Herstein [1], it is known that in a ring R in which given $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $x^n y = yx^n$, then either R is commutative or its commutator ideal is a nil ideal.

1. In this section we prove some general properties of a group G ∈ (C).

PROPOSITION 1. *Let* G ∈ (C). *Then*:

i) *every non trivial subgroup of* G *intersects every conjugate non trivially*;
ii) *every normal cyclic subgroup is in the center of* G.

Proof. *i*) Let $1 \neq A \leq G$. Suppose that for some $x \in G$

$$A \cap A^x = 1 . \tag{2}$$

Let $1 \neq a \in A$; then there exists a positive integer $t = t(a, x)$ such that

$$xa^t = a^t x ,$$

i.e.,

$$a^t = x^{-1} a^t x .$$

By (2)

$$a^t = x^{-1} a^t x = 1 .$$

Since G is torsion free, this implies $a = 1$, a contradiction.

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ii) Let $\langle a \rangle \trianglelefteq G$. For every $x \in G$ we have

$$x^{-1} ax = a^h$$

for some integer h. But for some positive integer k we have

$$xa^k = a^k x .$$

Thus $a^k = a^{hk}$. Since G is torsion free, this implies $h = 1$. Thus $ax = xa$ for every $x \in G$, i.e., $\langle a \rangle \leq Z(G)$, q.e.d.

PROPOSITION 2. *Let* $G \in (C)$ *and suppose* G *is finitely generated. Then*:

i) G *has a non trivial center*, $Z(G)$;
ii) $G/Z(G)$ *is a torsion group*;
iii) $G/Z(G)$ *has a trivial center*;
iv) *if* G *is solvable*, G *is Abelian.*

Proof. i) and *ii*). Let $G = \langle a_1, a_2, \cdots, a_r \rangle$ and let x be any element of G. Then

$$x^{n_i} a_i = a_i x^{n_i}$$

for some $n_i = n_i(x, a_i) > 0$, $i = 1, 2, \cdots, r$. If $n = \prod_{i=1}^{r} n_i$, then

$$x^n a_i = a_i x^n.$$

Thus x^n commutes with every generator of G and therefore $x^n \in Z(G)$. Since G is torsion free, $x^n \neq 1$ if $x \neq 1$. This proves *i*) and *ii*).

iii). Let $Z = Z(G)$ and $H/Z = Z(G/Z)$. If $H = Z$ there is nothing to prove. Suppose $H \neq Z$: we will show a contradiction. Let $x \in H - Z$. If $y^{-1} xy$ is any conjugate of x then $\overline{y^{-1}xy} = \bar{x}$ (mod Z), since $\bar{x}$ is central in $\bar{G} = G/Z$. Thus, $y^{-1} xy = xz$ for some $z \in Z$. If $z = 1$ for every $y \in G$, then $x = y^{-1} xy$, for all $y \in G$, i.e., $x \in Z$, a contradiction, since $x \in H - Z$. Thus, for some $y \in G$, $y^{-1} xy = xz$, with $z \neq 1$. By part *ii*), $x^n \in Z$ for some positive n, and therefore $y^{-1} x^n y = x^n$. Now

$$y^{-1} x^n y = (y^{-1} xy)^n = (xz)^n = x^n z^n ;$$

thus

$$x^n = x^n z^n$$

and this implies $z^n = 1$, with $z \neq 1$. Since G is torsion-free, this is a contradiction. This proves *iii*).

iv) Let $a, b \in G$. We want to prove that $ab = ba$. Consider $A = \langle a, b \rangle$, the subgroup generated by a and b. The group $\bar{A} = A/Z(A)$ is generated by the two elements $\bar{a}$ and $\bar{b}$ (mod $Z(A)$); by part *ii*), $\bar{A}$ is torsion. Thus $\bar{A}$ is a finitely generated solvable torsion group. It is well known tnat such groups are finite [2]. Thus $Z(A)$ has finite index in A. By a celebrated theorem of Schur [3], the derived group A' is finite. Since G is torsion-free, $A' = 1$, so that A is Abelian. Thus $ab = ba$, q.e.d.

COROLLARY. *Let* G $\in$ (C). *If* G *is locally solvable*, G *is Abelian.*

Proof. By Proposition 2, every finitely generated subgroup is abelian. Thus G is Abelian, q.e.d.

REMARK. The proof of part *iv*) of Proposition 2 shows that any condition that would ensure the answer in the affirmative to the Burnside problem would give the commutativity of G. For example, it is known that a torsion group of matrices over a field is locally finite [4]. Hence, in that situation—that is, for torsion-free groups of matrices—being in (C) is equivalent to being commutative. More generally, for groups embedded in rings satisfying a polynomial identity, since the Burnside problem has an affirmative answer there [5], our proposition also holds.

A conjecture: Let G be a torsion-free group. Consider the subgroup

$$M = \{x \in G \mid \forall g \in G\ \exists t = t(x, g) > 0 \text{ s.t. } xg^t = g^t x\}.$$

Clearly, M is a normal subgroup of G. We conjecture the following:

If G *is simple*, M = 1. In other words, a torsion-free simple group cannot belong to (C).

REFERENCES.

[1] I. N. HERSTEIN, *Two remarks on the commutativity of rings*, «Can. J. Math.», 7, 411–412 (1955).
[2] I. KAPLANSKY, *Fields and Rings*, The University of Chicago Press (1969), p. 103.
[3] W. R. SCOTT, *Group Theory*, Prentice Hall Inc. (1964), p. 433.
[4] I. N. HERSTEIN, *Noncommutative Rings*, Carus Math. Monographs, n. 15 (1968), p. 66.
[5] C. PROCESI, *On the Burnside problem*, «Journal of Algebra», 4, 421–426 (1966).