Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A new approach to the definition of topological degree for multi-valued mappings

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **47** (1969), n.6, p. 434–440.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1969_8_47_6_434_0>

Matematica. — *A new approach to the definition of topological degree for multi-valued mappings.* Nota di ARRIGO CELLINA e ANDRZEJ LASOTA presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si usa un teorema di approssimazione precedentemente dimostrato dal primo autore per ottenere una nuova dimostrazione del teorema antipodale per applicazioni multivoche e per definire il grado topologico per le stesse.

INTRODUCTION.

The approximation theorem proved in [1] permits us to obtain new simple proofs of fixed point theorems for multi-valued mappings. In this Note we shall show how it is possible to apply it to a larger class of problems for multi-valued mappings. We use it to get a new proof of the theorem on antipodes for multi-valued mappings [3], and we use it to define the topological degree for multi-valued mappings [2], [5]. The point is that this approach is quite elementary and does not require any knowledge of homology theory. Moreover we can immediately get these theorems for multi-valued mappings in metric locally convex spaces starting from the known theorems for single-valued mappings in finite dimensional spaces. An additional advantage of our approach is that the proof of the Antipodal Theorem for multi-valued mappings can be obtained independently of the definition and properties of topological degree. For this reason we present it at the beginning.

A drawback of the method is that it requires the mapping to be convex-valued and therefore does not yield the extension of the theory of topological degree to acyclic mappings defined on Euclidean space (as presented in [4]).

NOTATIONS AND BASIC DEFINITIONS.

If S is a metric space, $x, s \in S$, then $d(x, s)$ denotes the distance of x from s. If Z is also a metric space, $S \times Z$ is a metric space with $d((s, z), (x, y)) = \max\{d(s, x), d(z, y)\}$. For $A \subset S$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. The separation of A from B, $d^*(A, B)$ is defined to be $\sup\{d(x, B) : x \in A\}$. An open *ball* about x of radius $\varepsilon > 0$ is denoted by $B[x, \varepsilon]$. We also set $B[A, \varepsilon] = \{y \in S : d(y, A) < \varepsilon\}$. For A contained in the metric linear space Y, ∂A denotes the boundary of A, $\bar{A}$ its closure, and $\overline{co}A$ the *closed convex hull* of A.

2^Y is the set of subsets of Y, $K(Y)$ the set of *convex* subsets of Y and $CK(Y)$ the set of *closed convex* subsets of Y. A mapping $\Gamma : S \to 2^Y$ can be considered as a multi-valued mapping from S into Y. For $A \subset S$ we set

(*) Nella seduta del 13 dicembre 1969.

$\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$. The set $\Gamma(S)$ will be called the range of Γ and is denoted by $R(\Gamma)$. By the graph G of the mapping Γ we mean the subset of $S \times Y$ defined by

$$G = \{(s, y) : s \in S \quad \text{and} \quad y \in \Gamma(s)\} .$$

A mapping $\Gamma : S \to 2^Y$ is called *upper semi-continuous* (u.s.c.) at s if $\Gamma(s) \neq \varnothing$ and if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Gamma(B[s, \delta]) \subset B[\Gamma(s), \varepsilon]$. Γ is called u.s.c. on S if it is u.s.c. at each point $s \in S$.

A u.s.c. mapping $\Gamma : S \to 2^Y$ is called *compact* when its range is precompact, and it is called *finite dimensional* when its range is contained in a finite dimensional space.

The imbedding mapping from $S \subset X$ into X is denoted by i. A mapping $\Phi : S \to CK(X)$ of the form $\Phi = i - \Gamma$, where Γ is u.s.c. and compact (finite dimensional) is called a *compact (finite dimensional) vector field.*

The same definitions hold for single-valued mappings. For example, $\varphi : S \to X$ is a compact vector field when $f = i - \Phi$ is a continuous compact mapping.

BASIC APPROXIMATION THEOREM.

Our results are based on the following Approximation Theorem [1] and a simple Lemma.

THEOREM 1. *Let* X *be a metric space,* Y *a metric locally convex space,* $\Gamma : X \to CK(Y)$ *a u.s.c. mapping such that* $R(\Gamma)$ *is totally bounded. Then for* $\varepsilon > 0$ *arbitrary, there exists a continuous single-valued mapping* $f : X \to \overline{co}R(\Gamma)$ *depending on* ε*, such that*

$$d^*(F, G) < \varepsilon$$

where F *and* G *are the graphs of* f *and* Γ *respectively. Moreover the range of* f *is contained in a finite dimensional subspace of* Y.

LEMMA 1. *Let* X *and* Y *be metric spaces,* $\Gamma : X \to 2^Y$ *a u.s.c. multi-valued mapping and let* $f_n : X \to Y$ *be such that* $d^*(F_n, G) \to 0$ *where* F_n *and* G *are the graphs of* f_n *and* Γ *respectively. Then if* $(x_n, y_n) \subset F_n$ *and* $(x_n, y_n) \to (x_0, y_0)$,

$$(x_0, y_0) \subset G .$$

Since the convergence introduced in the preceding Lemma will often be used in what follows, we introduce the following definition.

Definition 1. Let X and Y be metric spaces, $\Gamma : X \to 2^Y$ a multi-valued mapping; we say that a sequence $\{\Gamma_n\}$ of multi-valued mappings from X into 2^Y converges to Γ (denoted by $\Gamma_n \to \Gamma$) when

$$d^*(G_n, G) \to 0$$

where G_n and G are the graphs of Γ_n and Γ. The same definition holds when Γ_n are single-valued.

THE ANTIPODAL THEOREM.

The following is the Antipodal Theorem for compact u.s.c. mappings in locally convex spaces. This theorem has been proved in [2] in Banach spaces.

THEOREM 1. *Let* X *be a metric locally convex space, with unit ball* B. *Let* $\Gamma : B \to CK(X)$ *be compact u.s.c. mapping. Set* $\Phi = i - \Gamma$ *and assume that*

$$\Phi(x) \cap \lambda\Phi(-x) = \varnothing$$

for all $0 \leq \lambda \leq 1$ *and all* $x \in \partial B$. *Then there exists a fixed point of* Γ *in* B, *i.e. for some* $x \in B$,

$$x \in \Gamma(x) .$$

Proof. By the Approximation Theorem, there exists a sequence $\{f_n\}$ of single-valued mappings $f_n : B \to X$, such that $f_n \to \Gamma$. Set $\varphi_n = i - f_n$. We claim that for all n sufficiently large, $\varphi_n(x) \neq \lambda\varphi_n(-x)$ for all $x \in \partial B$ and $0 \leq \lambda \leq 1$.

Suppose not. Then there exist sequences $\{x_n\} \subset \partial B$ and $\{\lambda_n\}$, $0 \leq \lambda_n \leq 1$, such that $\varphi_n(x_n) = \lambda_n \varphi_n(-x_n)$. It follows then that

$$(1 + \lambda_n) x_n = f_n(x_n) - \lambda_n f_n(-x_n)$$

Since

$$f_n(x_n) \in \overline{co}R(\Gamma) \quad , \quad f_n(-x_n) \in \overline{co}R(\Gamma)$$

and $\overline{co}R(\Gamma)$ is compact by a theorem of Mazur, we can assume (taking subsequences if necessary) that there exist $y_0, y_1 \in \overline{co}R(\Gamma)$ such that $f_n(x_n) \to y_0$ and $f_n(-x_n) \to y_1$. We can also assume that $\lambda_n \to \lambda_0$. Therefore there exists $x_0 \in \partial B$, such that $(1 + \lambda_n) x_n \to (1 + \lambda_0) x_0$. By Lemma 1 it follows that $y_0 \in \Gamma(x_0)$ and $y_1 \in \Gamma(-x_0)$, so that $(x_0 - y_0) \in (x_0 - \Gamma(x_0))$ and $\lambda_0(-x_0 - y_1) \in \in \lambda_0(-x_0 - \Gamma(-x_0))$. On the other hand $(x_0 - y_0) = \lambda_0(-x_0 - y_1)$, contradicting the hypothesis.

Each f_n has a finite dimensional range. Consider its intersection with B. By Borsuk's Theorem, each f_n has a fixed point, say $y_n \in B$. Since each fixed point belongs to the compact set $\overline{co}R(\Gamma)$, we can take a converging subsequence $y_{k_n} \to y_0$. Again, since $(y_{k_n}, y_{k_n}) \in F_{k_n}$ (F_n stands for the graph of f_n) and $(y_{k_n}, y_{k_n}) \to (y_0, y_0)$, by Lemma 1 it follows that $(y_0, y_0) \in G$, i.e. $y_0 \in \Gamma(y_0)$.

DEFINITION OF TOPOLOGICAL DEGREE.

Let X be a metric locally convex space and D an open, bounded (with respect to the metric d of X) subset of X. Let $\varphi : \bar{D} \to X$ be a finite dimensional vector field. For any finite dimensional subspace $Y \subset X$ containing the range of $f = i - \varphi$, the topological degree of $\varphi|_Y$ (the restriction of φ to Y) at a point $p \in Y \setminus \varphi(\partial D)$, is meaningful. We denote this degree by

$$\deg(p, \varphi|_Y, D \cap Y). \tag{1}$$

From known properties of the topological degree in finite dimensional spaces, it follows that (1) does not depend on the choice of Y. Therefore we can define the topological degree of the finite dimensional vector field $\varphi : \bar{D} \to X$ at the point p, setting

$$\deg(p, \varphi, D) = \deg(p, \varphi|_Y, D \cap Y).$$

Definition. Let $\Phi : \bar{D} \to CK(X)$ be a compact vector field and let $p \notin \Phi(\partial D)$. Let $\{\varphi_n\}$ be a sequence of single-valued finite dimensional vector fields converging to Φ. We define the *topological degree* of Φ to be

$$\deg(p, \Phi, D) = \lim_{n \to \infty} \deg(p, \varphi_n, D). \tag{2}$$

For the preceding definition to make sense, we have to show that:

(*i*) given any such Φ, there exists a sequence of single-valued, finite dimensional vector fields φ_n converging to Φ;

(*ii*) for n sufficiently large, $p \notin \varphi_n(\partial D)$, so that $\deg(p, \varphi_n, D)$ is defined;

(*iii*) the limit of the right hand side of (2) exists and does not depend on the choice of the sequence.

Proof of (i). Set $\Phi = i - \Gamma$. By Theorem 1 there exists a sequence of finite dimensional mappings $\{f_n\}$ $(f_n : \bar{D} \to \overline{co}R(\Gamma))$ converging to Γ. It is easy to see then that $\varphi_n = i - f_n$ converges to Φ.

Proof of (ii). Suppose the claim false, then there exists a sequence of integers $\{k_n\}$ such that $p = x_{k_n} - f_{k_n}(x_{k_n})$ for some $x_{k_n} \in \partial D$. It is easy to show that the compactness of $\overline{R(\Gamma)}$ yields the compactness of $\overline{\{f_{k_n}(x_{k_n})\}}$ Taking a subsequence if necessary, we can assume that $f_{k_n}(x_{k_n})$ converges to a point y_0. Then x_{k_n} converges to $x_0 \stackrel{\text{def}}{=} y_0 + p$. By Lemma 1 it follows that

$$y_0 = x_0 - p \in \Gamma(x_0), \qquad x_0 \in S$$

contradicting the assumption.

Proof of (iii). To prove the claim we are going to show that for any given sequence $\{\varphi_n\}$ converging to Φ and n and m sufficiently large,

$$H_{n,m}(t, x) = t\varphi_n(x) + (1 - t)\varphi_m(x) \neq p$$

for all $t \in [0, 1]$ and all $x \in \partial D$.

Suppose not. Then there exist two sequences of integers, $\{k_n\}$ and $\{l_n\}$, a sequence of real numbers $\{t_n\} \subset [0, 1]$ and a sequence of points $\{x_n\} \subset \partial D$ such that

$$t_n \varphi_{k_n}(x_n) + (1 - t_n)\varphi_{l_n}(x_n) = p.$$

Therefore

$$t_n f_{k_n}(x_n) + (1 - t_n) f_{l_n}(x_n) = x_n - p .$$

In virtue of the compactness of $\overline{co}\mathrm{R}(\Gamma)$, taking subsequences if necessary, we can assume that $t_n \to t_0$, $f_{k_n}(x_n) \to y_1$ and $f_{l_n}(x_n) \to y_2$, for some t_0, y_1 and y_2. It follows then that $x_n \to x_0$ for some $x_0 \in \partial\mathrm{D}$. Therefore

$$t_0 y_1 + (1 - t_0) y_2 = x_0 - p$$

or

$$t_0 (x_0 - y_1) + (1 - t_0)(x_0 - y_2) = p .$$

Since

$$t_0 (x_0 - y_1) + (1 - t_0)(x_0 - y_2) \in t_0 (i - \Gamma)(x_0) + (1 - t_0)(i - \Gamma)(x_0)$$

we have

$$p \in (i - \Gamma)(x_0) \quad , \quad x_0 \in \partial\mathrm{D}$$

contradicting the hypothesis.

Properties of topological degree.

In the following X is a metric locally convex space; D, D_i are open and bounded subsets of X; Φ and Φ_i are compact vector fields defined on $\bar{\mathrm{D}}$ with values in CK (X).

Proposition 1. If $\Phi_n \to \Phi$, and $p \in \Phi(\partial\mathrm{D})$, then

$$\deg(p, \Phi_n, \mathrm{D}) \to \deg(p, \Phi, \mathrm{D}) .$$

Proof. It is easy to verify that for sufficiently large n, $p \notin \Phi_n(\partial\mathrm{D})$, so that $\deg(p, \Phi_n, \mathrm{D})$ is meaningful. For each n let $\{\varphi_{ni}\}$ be a sequence of single-valued, finite dimensional vector fields, such that $\varphi_{ni} \to \Phi_n$ $(i \to \infty)$. Choosing subsequences if necessary, we can assume that $\deg(p, \varphi_{ni}, \mathrm{D}) = \deg(p, \Phi_n, \mathrm{D})$ for $i \geq n$. It is easy to check that $\varphi_{nn} \to \Phi$ and consequently

$$\deg(p, \Phi, \mathrm{D}) = \lim_{n \to \infty} \deg(p, \varphi_{nn}, \mathrm{D}) = \lim_{n \to \infty} \deg(\mathrm{P}, \Phi_n, \mathrm{D}) .$$

COROLLARY. *Let* $\varphi : \bar{\mathrm{D}} \to \mathrm{X}$ *be a single-valued vector field such that* $\varphi(x) \in \Phi(x)$ *for all* $x \in \mathrm{D}$. *Then*

$$\deg(p, \varphi, \mathrm{D}) = \deg(p, \Phi, \mathrm{D})$$

whenever the right-hand side is meaningful.

Proposition 2. Let Φ_0, Φ_1 be homotopic avoiding the point p, i.e. there is a family Φ_t $(t \in [0, 1])$ of compact vector fields which depend continuously on t ($\Phi_{t_n} \to \Phi_t$ in the sense of Definition 1, when $t_n \to t$) such that

$$p \notin \Phi_t(\partial D) \quad , \quad t \in [0, 1] .$$

Then

$$\deg(p, \Phi_0, D) = \deg(p, \Phi_1, D) . \tag{3}$$

Proof. From Proposition 1 it follows that the function $\delta(t) \overset{\text{def}}{=} \deg(p, \Phi_t, D)$ is continuous. Since the range of δ is discrete (the integers) and $[0, 1]$ is connected, then $\delta(t)$ must be constant.

The preceding proposition can be formulated in a stronger form using a different definition of homotopy [1], [3].

Proposition 2'. Let $H = i - \Gamma$ and let $\Gamma : [0, 1] \times \bar{D} \to CK(X)$ be a u.s.c. and compact mapping. Assume that $H(0, \cdot) = \Phi_1(\cdot)$, $H[1, \cdot] = \Phi_2(\cdot)$ and $p \notin H(t, \partial D)$ for $t \in [0, 1]$. Then (3) holds.

Proof. By Theorem 1, there exists a sequence $\{f_n\}$ of finite dimensional mappings converging to Γ. By the usual argument we can show that for n sufficiently large, $p \neq x - f_n(t, x)$ for all $t \in [0, 1]$ and all $x \in \partial D$. Set $\varphi_n^1 = i - f_n(0, \cdot)$ and $\varphi_n^2 = i - f_n(1, \cdot)$. Then $\deg(p, \varphi_n^1, D) = \deg(p, \varphi_n^2, D)$. Moreover $\varphi_n^1 \to \Phi_1$ and $\varphi_n^2 \to \Phi_2$ and by Proposition 1 the assertion follows.

Proposition 3. If $D = \bigcup_{i=1}^{n} D_i$, D_i are disjoint, and $\partial D_i \subset \partial D$, then for every $p \notin \Phi(\partial D)$,

$$\deg(p, \Phi, D) = \sum_{i=1}^{n} \deg(p, \Phi, D_i) .$$

Proof. Since the result is true for single-valued compact vector fields, the proof follows from the usual convergence argument.

The same argument applies to the proof of the following

Proposition 4. If p and q belong to the same component of $R^n \setminus \varphi(\partial D)$, then

$$\deg(p, \Phi, D) = \deg(q, \Phi, D) .$$

Proposition 5. If $\deg(p, \varphi, D) \neq 0$, there exists $x_0 \in \bar{D}$ such that $p \in \Phi(x_0)$.

Proof. By definition of topological degree there exists a sequence of φ_n (single-valued) such that $\varphi_n \to \Phi$ and $\deg(p, \varphi_n, D) \to \deg(p, \Phi, D)$. Therefore for each sufficiently large n, there exists x_n such that $p = \varphi_n(x_n)$. Since the range of $i - \Phi$ is precompact we can assume, passing to subsequences if necessary, that the sequence $x_n - \varphi(x_n)$ is convergent. Therefore x_n is convergent too. Set $x_0 = \lim x_n$. By Lemma 1 it follows that $p \in \Phi(x_0)$ which completes the proof.

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