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Arrigo Cellina

A theorem on the approximation of compact multi-valued mappings

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RENDICONTI

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DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A theorem on the approximation of compact multi-valued mappings.* Nota di ARRIGO CELLINA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si presenta un teorema di approssimazione mediante applicazioni univoche, per applicazioni multivoche semicontinue superiormente, il cui codominio sia contenuto in un insieme compatto. Si dimostra poi un teorema di estensione per omotopia.

INTRODUCTION.

In [1] it was proved that given an upper semicontinuous point-to-convex valued mapping Γ, defined on a compact subset K of a metric space X, it is always possible to "approximate" it by continuous single-valued mappings. Precisely it was proved that for any $\varepsilon > 0$ there exists a continuous single-valued function f, defined on K, such that $d^*(F, G) < \varepsilon$, where F and G are the graphs of f and Γ, and d^* means the separation of F from G (cfr. the definition below). As an application of this theorem, a simple proof of Kakutani Fixed Point Theorem was presented.

Purpose of the present Note is to show that an analogous theorem holds for *compact* multi-valued mappings defined over an arbitrary metric space. Neither of these results contains the other: in fact, in the theorem we are going to prove, the domain of definition of Γ need not be compact, and in the theorem proved in [1] images of points were not required to be compact (nor even closed) and therefore the range of Γ did not need to be compact.

(*) Nella seduta del 13 dicembre 1969.

As applications of our main theorem we shall present an extension theorem, analogous to Dugundji's Theorem for single-valued mappings, and a homotopy extension theorem for multi-valued mappings.

NOTATIONS AND BASIC DEFINITIONS.

If S is a metric space, $x , s \in S$, then $d(x , s)$ denotes the distance of x from s. If Z is also a metric space, $S \times Z$ is a metric space with $d((s , z) , (x , y)) = \max \{d(s , x) , d(z , y)\}$. For $A \subset S$, $d(x , A) = \inf \{d(x , y) : y \in A\}$. The separation of A from B, $d^*(A , B)$ is defined to be $\sup \{d(x , B) : x \in A\}$. An open *ball* about x of radius $\varepsilon > 0$ is denoted by $B[x , \varepsilon]$. We also set $B[A , \varepsilon] = \{y \in S : d(y , A) < \varepsilon\}$. For A contained in the metric linear space Y, ∂A denotes the boundary of A, $\bar{A}$ its closure, and $\overline{co} A$ the *closed convex hull* of A.

2^Y is the set of subsets of Y, K(Y) the set of *convex* subsets of Y and CK(Y) the set of *closed convex* subsets of Y. A mapping $\Gamma : S \to 2^Y$ can be considered as a multi-valued mapping from S into Y. For $A \subset S$ we set $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$. The set $\Gamma(S)$ will be called the range of Γ and is denoted by $R(\Gamma)$. By the graph G of the mapping Γ we mean the subset of $S \times Y$ defined by

$$G = \{(s , y) : s \in S \quad \text{and} \quad y \in \Gamma(s)\} .$$

A mapping $\Gamma : S \to 2^Y$ is called *upper semi-continuous* (u.s.c.) at s if $\Gamma(s) \neq \varnothing$ and if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Gamma(B[s , \delta]) \subset B[\Gamma(s) , \varepsilon]$. Γ is called u.s.c. on S if it is u.s.c. at each point $s \in S$.

A u.s.c. mapping $\Gamma : S \to 2^Y$ is called *compact* when its range is pre-compact, and it is called *finite dimensional* when its range is contained in a finite dimensional space.

RESULTS.

The following is the Approximation Theorem for compact multivalued mappings with images in a metric locally convex space. In its proof we shall assume that the metric has been chosen so that balls are convex.

THEOREM 1. *Let* X *be a metric space,* Y *a metric locally convex space,* $\Gamma : X \to CK(Y)$ *a u.s.c. mapping such that* $R(\Gamma)$ *is totally bounded. Then for* $\varepsilon > 0$ *arbitrary, there exists a continuous single-valued mapping* $f : X \to co\, R(\Gamma) \cap B[R(\Gamma) , \varepsilon]$, *depending on* ε, *such that*

$$d^*(F , G) < \varepsilon$$

where F *and* G *are the graphs of* f *and* Γ.

Proof. Let

$$\rho(x , \varepsilon) = \sup \{\delta < \varepsilon : \exists\, x' \in B[x , \varepsilon] \ni \Gamma(B[x , \varepsilon]) \subset B[\Gamma(x') , \varepsilon/2]\} .$$

Since Γ is u.s.c., $\rho(x , \varepsilon) > 0$ for all $x \in X$. We are going to prove that $\rho(\cdot , \varepsilon)$ is a lower semi-continuous function from X to the positive reals.

Let $x^0 \in X$. We claim that

$$\rho^0 \overset{\text{def}}{=} \rho(x^0, \varepsilon) \leq \overline{\lim_{x \to x^0}} \rho(x, \varepsilon)$$

Suppose not. Then there exists a sequence $\{x_n\} \subset X$, $x_n \to x^0$, such that $n \geq N$ implies

$$\rho_n \overset{\text{def}}{=} \rho(x_n, \varepsilon) < \rho^0 - \eta,$$

for some $\eta > 0$. For a fixed ζ, $0 < \zeta < \eta$, there exists x'_0, $d(x^0, x'_0) = \varepsilon - \varepsilon_1$, $\varepsilon_1 > 0$, such that

$$\Gamma(B[x^0, \rho^0 - \zeta]) \subset B[\Gamma(x'_0), \varepsilon/2].$$

Let $\eta_1 < \varepsilon_1$ and $M > N$ such that $n > M$ implies $d(x_n, x^0) < \eta_1$. Then

$$\Gamma(B[x_n, \rho^0 - \zeta - \eta_1]) \subset \Gamma(B[x^0, \rho^0 - \zeta]) \subset B[\Gamma(x'), \varepsilon/2]$$

and $d(x_n, x') < \varepsilon$. Therefore $n \geq M$ implies $\rho_n \geq \rho^0 - \zeta - \eta_1$ i.e. $\overline{\lim_{n\to\infty}} \rho_n \geq \rho_n - \zeta - \eta_1$. Since η_1 was arbitrary,

$$\overline{\lim_{n\to\infty}} \rho_n \geq \rho^0 - \zeta$$

contradicting the hypothesis. (An analogous reasoning would show that ρ is continuous).

Since $R(\Gamma)$ is totally bounded, for a fixed $\varepsilon > 0$ there exists a finite number of points $y_i \in R(\Gamma)$ such that $R(\Gamma) \in \cup B[y_i, \varepsilon/2]$. Let us define the sets O_i by

$$O_i = \{x \in X : B[y_i, \varepsilon/2] \cap \Gamma(B[x, \rho(x, \varepsilon)/2]) \neq \varnothing\}.$$

We claim that the O_i are open. Let $x \in O_i$. Then there exists x', $d(x, x') = \rho(x, \varepsilon)/2 - \eta_i$, $\eta_i > 0$, such that $\Gamma(x') \cap B[y_i, \varepsilon/2] \neq \varnothing$. Since ρ is lower semi-continuous, there exists a $\delta > 0$ such that $\rho(\xi, \varepsilon)/2 > \rho(x, \varepsilon)/2 - \eta_i/2$ for every $\xi \in B[x, \delta]$. It follows then that when $d(\xi, x) < \min\{\delta, \eta_i/2\}$, $x' \in B[\xi, \rho(\xi, \varepsilon)/2]$ and therefore $\xi \in O_i$, i.e. O_i is open.

It is clear that $X \subset \{O_i\}_{i=1}^r$. Let $\{p_i\}_{i=1}^r$ be a partition of unity subordinated to $\{O_i\}_{i=1}^r$ and set

$$f(x) = \sum_{i=1}^{r} p_i(x)\, y_i.$$

The function $f: X \to Y$ is continuous and for each fixed x, $f(x)$ is a convex combination of some y_i, $y_i \in B[\Gamma(B[x, \rho(x, \varepsilon)/2]), \varepsilon/2]$. From 2.2 it follows that there exists an x', $d(x, x') < \varepsilon$, such that $\Gamma(B[x, \rho(x, \varepsilon)/2]) \subset B[\Gamma(x'), \varepsilon/2]$ and so $y_i \in B[\Gamma(x'), \varepsilon]$. Since $\Gamma(x')$ is convex, also $f(x) \in B[\Gamma(x'), \varepsilon]$. Finally

$$d((x, f(x)), G) \leq \max\{d(x, x'), d(f(x), \Gamma(x'))\} < \varepsilon$$

i.e. $d^*(F, G) < \varepsilon$.

Remark 1. The approximating functions are finite dimensional compact mappings.

THEOREM 2. *Let* X *be a metric space and* Y *be a metric locally convex space. Let* $\Gamma : A \to CK(Y)$ *be a u.s.c. compact mapping defined on the closed set* $A \subset X$. *Then* Γ *can be extended to a compact u.s.c. mapping* $\tilde{\Gamma} : X \to CK(Y)$ *such that* $R(\tilde{\Gamma}) \subset \overline{co}R(\Gamma)$.

Proof. Let $\varepsilon_\nu \downarrow 0$. Following Theorem 2.1, let $f_\nu : A \to coR(\Gamma)$ such that $d^*(F_\nu, G) < \varepsilon_\nu$, where F_ν and G are the graphs of f_ν and Γ; f_ν are compact mappings. By Dugundji's theorem, each f_ν can be extended to a $\tilde{f}_\nu : X \to \overline{co}R(\Gamma)$.

Let $x \in X$ and consider the set

$$\Delta(x) = \{y : \exists \{n_\nu\} : \tilde{f}_{n_\nu}(x) \to y\}.$$

Since $\overline{co}R(\Gamma)$ is compact by a theorem of Mazur, $\Delta(x) \neq \varnothing$ for all $x \in X$. Moreover we have that for $x \in A$, $\Delta(x) \subset \Gamma(x)$: in fact suppose not. Then there exists a $y^* \in \Delta(x)$, $y^* \notin \Gamma(x)$. Since G is closed, $d((x, y^*), G) = \eta > 0$. Since for n sufficiently large the graps of f_n are contained in a neighborhood of G of radius $\eta/2$, this contradicts the existence of a subsequence $f_{n_\nu}(x) \to y^*$.

Let D be the graph of the mapping $x \to \Delta(x)$ and $\bar{D}$ its closure (in $X \times Y$). Since G is closed, we have that in $A \times Y$, $\bar{D} \subset G$. Let $\bar{\Delta} : X \to 2^Y$ be the mapping whose graph is $\bar{D}$. Then $\bar{\Delta}$ is a closed multi-valued mapping and since it range is contained in the compact set $\overline{co}R(\Gamma)$, it is u.s.c. and compact. The mapping Γ defined by

$$\tilde{\Gamma}(x) = \begin{cases} \Gamma(x) & \text{for } x \in A \\ co\bar{\Delta}(x) & \text{for } x \in X \setminus A \end{cases}$$

is clearly u.s.c. and is the required extension.

The following theorem is the Homotopy Extension Theorem for multi-valued compact vector fields. We define a multi-valued compact vector field to be a mapping Φ that can be represented in the form

$$\Phi(x) = x - \Gamma(x)$$

where Γ is a compact multi-valued mapping. A compact multi-valued vector field Φ is said to be singularity free when $0 \notin R(\Phi)$. Two compact u.s.c. vector fields $\Phi_1(x) = x - \Gamma_1(x)$ and $\Phi_2(x) = x - \Gamma_2(x)$, where $\Gamma_i : S \to CK(Y \setminus 0)$, $i = 1, 2$, are called homotopic when there is a u.s.c. compact multi-valued mapping $H : S \times [0, 1] \to CK(Y \setminus 0)$ such that

$$H(0, x) = \Gamma_1(x) \qquad \text{and} \qquad H(1, x) = \Gamma_2(x).$$

THEOREM 2.4. *Let* Y *be a metric locally convex space,* $X \subset Y$ *and* X^0 *a closed subset of* X. *Let* $\Phi_1, \Phi_2 : X^0 \to CK(Y \setminus 0)$ *be two homotopic singularity free compact u.s.c. vector fields. If there exists an extension* $\tilde{\Phi}_1 : X \to CK(Y \setminus 0)$ *of* Φ_1 *over* X, *there exists also an extension* $\tilde{\Phi}_2 : X \to CK(Y \setminus 0)$ *of* Φ_2 *over* X *such that* $\tilde{\Phi}_1$ *and* $\tilde{\Phi}_2$ *are homotopic as mappings from* X *into* $CK(Y \setminus 0)$.

Proof. Using Theorem 2.3 the proof follows with minor modifications the one presented in [3] for single-valued compact fields.

Applications of Theorem 1 will be presented in [2]. An application of Theorem 3 to the existence of critical sets for equations of evolution has been presented in [4].

REFERENCES.

[1] A. CELLINA, *Approximation of set-valued functions and fixed point theorems*, « Ann. Mat. Pura Appl. », *82*, pp. 17–24 (1969).

[2] A. CELLINA and A. LASOTA, *A new approach to the definition of topological degree for multi-valued mappings*, this issue.

[3] A. GRANAS, *The theory of compact vector fields amd some applications to topology of functional spaces*, « Rozprawy Matematyczne », *30* (1962).

[4] G. S. JONES, *On Critical Sets for Nonlinear Equations of Evolution with Non-uniqueness*, University of Maryland Report BN 573 (1968).