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The use of the epipolar axis in the analytical relative orientation

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Fotogrammetria. — *The use of the epipolar axis in the analytical relative orientation.* Nota di GIOVANNA TOGLIATTI, presentata (*) dal Socio L. SOLAINI.

RIASSUNTO. — Il presente studio si propone di svincolare l'orientamento relativo di due fotogrammi dai parametri tradizionali, introducendo invece come incognite 2 coppie di coordinate polari che definiscono la posizione delle rette nucleari rispetto alle due stelle di raggi proiettanti.

La conoscenza di tali elementi consente il calcolo dei punti del modello. In una successiva fase sperimentale si vedrà se questo metodo, concettualmente del tutto diverso dall'usuale, permette di evitare gli inconvenienti che la forte correlazione esistente tra i tradizionali elementi dell'orientamento relativo causa nella determinazione dello stesso e nella formazione del modello.

1. – The present paper has been suggested by the shortcomings that have been noticed in the traditional method of relative orientation, which achieves the co-planarity of all the pairs of homologous projecting rays by means of the determination of five parameters, either three rotations of one camera and two of the other, or two translations and three rotations of the second, the first one being held fixed. Some of these parameters are strongly correlated; to this cause the fact has been ascribed that small errors in the measure of plate coordinates may influence the final results of the orientation which oscillate within limits much wider than those consistent with the size of the measure errors. In other words, even if the residual parallaxes are small, the orientation parameters are affected by non negligeable errors and the models show deformations.

Therefore, it has been tried to disengage the determination of relative orientation from the traditional parameters, making use of the epipolar axis, that is of the straight line connecting the taking points, which is the axis of the sheaf of planes containing all pairs of homologous rays. This element, essential in the study of the geometrical properties of relative orientation, had been repeatedly mentioned by Finsterwalder and other pioneers of photogrammetry; however it has been left off and there is no record that its characteristics have been exploited.

The unknowns assumed in the proposed method, which is applicable to analytical photogrammetry only, are the two pairs of parameters giving the direction of the epipolar axis within the bundles of homologous projecting rays of both photograms; the fifth unknown, that is the rotation of one camera around the epipolar axis, will, as we shall see later on, authomatically follow. In this paper the fundamental formulae are given, which will be experimentally applied in the future.

(*) Nella seduta del 15 novembre 1969.

2. – The problem can be set up in the following terms: given five projecting rays r_i coming out of the taking point O of the left camera and the corresponding rays r'_i coming out of O′ of the right camera, one must determine a straight line n through O and a straight line n' through O′ so that the two sets of five planes nr_i and $n'r'_i$ may be superposed, that is to say that the dihedral angle determined by any two nr_i planes be equal to the dihedral angle determined by the two corresponding $n'r'_i$ planes. Such straight lines n, n' will be called epipolar axes.

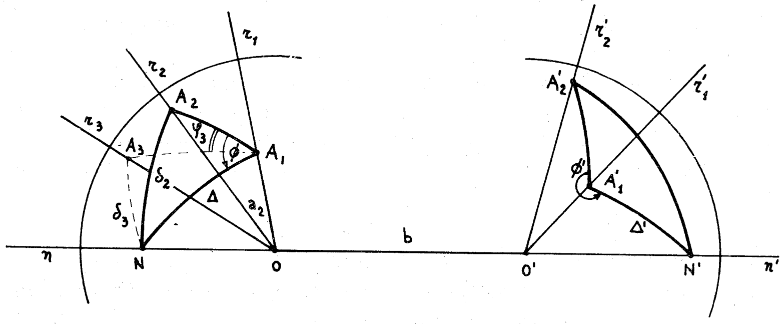

Fig. 1.

In order to make the exposure easier, it is expedient to make use of two direction spheres, one with centre in O, the other in O′, which intersect the two bundles of rays (see fig. 1). Five points A_i are, therefore, obtained in the first sphere and five points A'_i on the second one, all in a generical position. The problem is now to determine on both spheres the two epipoles N, N′, so that the spherical angles $A_i NA_j$ and $A'_i N' A'_j$ be respectively equal for any pair of i, j. It is obvious that it will be sufficient to obtain that the four angles $A_1 NA_j$ $(j = 2, 3, 4, 5)$ be equal to the corresponding ones, being A_1 anyone of the five points A_i.

Let now one set on the first sphere a polar coordinate system with origin in A_1 and the polar axis on $A_1 A_2$, oriented from A_1 to A_2; let then be

$$\begin{aligned} A_1 A_i &= a_i & & \\ A_2 A_1 A_i &= \varphi_i & NA_i &= \delta_i & (i = 2, 3, 4, 5) \\ A_2 A_1 N &= \Phi & NA_1 &= \Delta ; & \end{aligned}$$

similarly on the second sphere.

As it will be shown later on, the a_i and φ_i can be deduced from the measure of the plate coordinates of the homologous points and from the knowledge of the camera interior orientation, whereas the unknowns of the problem are the polar coordinates Δ, Φ of N and Δ', Φ' of N′.

Considering in the first place the spherical triangle $A_1 A_2 N$, its angle in N can be expressed as a function of the two sides a_2, Δ and of the angle Φ which they bound, as follows:

$$\operatorname{cotg} A_1 NA_2 = \frac{\operatorname{cotg} a_2 \operatorname{sen} \Delta - \cos \Delta \cos \Phi}{\operatorname{sen} \Phi} .$$

An analogous computation can be made for the triangle $A'_1 A'_2 N'$ and, since the two angles in N, N' must be equal, the following equation in Δ, Φ, Δ', Φ' can be set.

$$\text{(1)} \qquad \frac{\operatorname{cotg} a_2 \operatorname{sen} \Delta - \cos \Delta \cos \Phi}{\operatorname{sen} \Phi} = \frac{\operatorname{cotg} a'_2 \operatorname{sen} \Delta' - \cos \Delta' \cos \Phi'}{\operatorname{sen} \Phi'} .$$

Considering now the triangles $A_1 A_i N$, $A'_1 A'_i N'$ $(i = 3, 4, 5)$, their angles in N, N' can be computed as previously seen, after substituting a_i to a_2 and $\Phi - \varphi_i$ to Φ; the three following equations will then be set:

$$\text{(2)} \qquad \frac{\operatorname{cotg} a_i \operatorname{sen} \Delta - \cos \Delta \cos (\Phi - \varphi_i)}{\operatorname{sen} (\Phi - \varphi_i)} = \frac{\operatorname{cotg} a'_i \operatorname{sen} \Delta' - \cos \Delta' \cos (\Phi' - \varphi'_i)}{\operatorname{sen} (\Phi' - \varphi'_i)}$$

$$(i = 3, 4, 5)$$

which, with the 1), constitute a set of four independent equations in the four previously mentioned unknowns. This set of equations can be solved by means of an iterative method, but must be first linearized by means of the approximate values of the unknowns, which, in all cases, can be easily evaluated. It is obvious that the five points A_i are those strictly necessary, whereas a redundant number of homologous points can be used, thereby finding the values of the unknowns by means of the least squares method. In this case, attention will have to be paid to the fact that equations of type (1), (2) are not independent, since, as we shall see later on, the computation of *all* a_i and φ_i makes use of the plate coordinates of A_1, the errors of which reflect themselves on all the observation equations.

Having determined Φ, Δ, Φ', Δ', the spherical distances δ, δ', relative to any pair of homologous rays, represented on the direction spheres by the generical points $A(a, \varphi)$, $A'(a', \varphi')$ can be computed from the triangles $A_1 AN$, $A'_1 A' N'$ with the formula:

$$\text{(3)} \qquad \cos \delta = \cos a \cos \Delta + \operatorname{sen} a \operatorname{sen} \Delta \cos (\Phi - \varphi)$$

and analogously for δ'.

Summarizing, if the angles a_i, a'_i $(i = 2, 3, 4, 5)$ and φ_i, φ'_i $(i = 3, 4, 5)$, relative to whatever five pairs of homologous rays are known, the equations (1), (2) determine the epipoles and the formulae (3) give their distance both from the points A_i, A'_i $(i = 2, 3, 4, 5)$ and from any other pair of homologous points for which the values of a, a', φ, φ', had been previously computed.

3. – Reverting to the two sets of five projecting rays r_i, r'_i, we can now regard the two epipolar axes as known. There is still one operation to be done, i.e. the movement, for instance, of the second bundle of five rays, so that the planes $n' r'_i$ may coincide respectively with the planes $n r_i$. Let one place point O' in an arbitrary point of the straight line ON at a distance b from O and the epipolar axis $O' N'$ along the straight line ON (see fig. 2).

Now, for the definition itself of the epipolar axes and for the manner in which they have been analytically determined, if the second photogram is rotated around O′ N′ so that the projecting ray r'_i lay in the same plane with r_i, also all the other pairs of homologous rays will lay in the same plane. In fact the dihedral angles between any two pairs of planes nr_i, nr_j are equal to the corresponding dihedral angles between planes $n' r'_i$, $n' r'_j$.

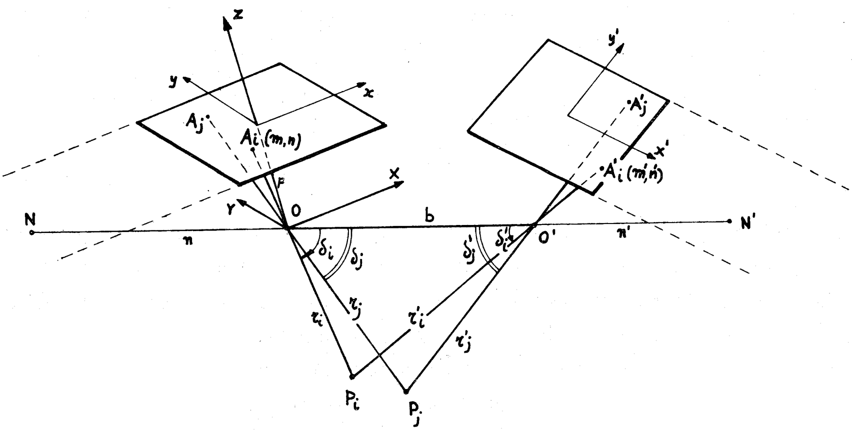

Fig. 2.

Each projecting ray r_i will intersect r'_i in a point P_i which represents a point of the model. In order to compute its coordinates, it is suitable to establish two terns of cartesian orthogonal axes, with origin in each one of the taking points, with Z axes along the camera axes turned towards the plates, X and Y axes parallel to the x , y axes which can be defined on the plates by means of the fiducial marks. The coordinates of the model points will be referred to any one of the two terns, for instance to the one whose origin is in O.

Let then be A_i , A'_i, of plate coordinates (m_i , n_i) , (m'_i , n'_i), the images of P_i on the two plates, and p the principal distance. In the triangle OO′ P_i of fig. 2 it is

$$\text{(4)} \qquad OP_i = \operatorname{sen} \delta'_i \cdot \frac{b}{\operatorname{sen} (\delta_i + \delta'_i)} \cdot$$

On the other hand, being (m_i , n_i , p) the coordinates of A_i , the straight line r_i has the following parametrical equations:

$$\text{(5)} \qquad X = m_i\, t \qquad Y = n_i\, t \qquad Z = pt.$$

The point P_i of such straight line will correspond to a value of t such that the distance OP_i be equal to the expression given by (4). That is:

$$- t \sqrt{m_i^2 + n_i^2 + p^2} = b \cdot \frac{\operatorname{sen} \delta'_i}{\operatorname{sen} (\delta_i + \delta'_i)} \cdot$$

Solving for t and substituting in (5), the following coordinates of P_i are obtained:

$$
\begin{aligned}
X_i &= \frac{-m_i}{\sqrt{m_i^2+n_i^2+p^2}} \cdot \frac{b \operatorname{sen} \delta'_i}{\operatorname{sen}(\delta_i+\delta'_i)} \\
Y_i &= \frac{-n_i}{\sqrt{m_i^2+n_i^2+p^2}} \cdot \frac{b \operatorname{sen} \delta'_i}{\operatorname{sen}(\delta_i+\delta'_i)} \qquad (6) \\
Z_i &= \frac{-p}{\sqrt{m_i^2+n_i^2+p^2}} \cdot \frac{b \operatorname{sen} \delta'_i}{\operatorname{sen}(\delta_i+\delta'_i)} \,.
\end{aligned}
$$

An analogous computation can be repeated for all those points of the model whose plate coordinates have been measured on a comparator. The model obtained in such a way will then be scaled and absolutely oriented by means of the usual analytical procedures.

4. – The data to be introduced in formulae (1), (2), (3), that is to say the a and φ, can be obtained as follows starting from the plate coordinates of the homologous points.

The coordinates of A_1 and of a generical A_i be respectively (m_1, n_1, p) and (m_i, n_i, p) with reference to the previously mentioned cartesian tern with origin in O. The equations of the straight lines OA_1 and OA_i are:

$$
\frac{X}{m_1} = \frac{Y}{n_1} = \frac{Z}{p} \quad ; \quad \frac{X}{m_i} = \frac{Y}{n_i} = \frac{Z}{p}
$$

wherefrom the angles between the two straight lines OA_1, OA_i:

$$
\cos a_i = \frac{m_1 m_i + n_1 n_i + p^2}{\sqrt{(m_1^2+n_1^2+p^2)(m_i^2+n_i^2+p^2)}} \,. \tag{7}
$$

The computation of the angles a' will be carried out in the same way by means of the plate coordinates of points A'_1, A'_i with reference to the tern of axes with origin in O′.

Finally, one must compute the angle φ, that is, for instance, the dihedral angle between plane $r_1 r_2 = OA_1 A_2$ and plane $r_1 r_i = OA_1 A_i$. The equations of such planes are respectively:

$$
\begin{aligned}
p(n_1 - n_2) X + p(m_2 - m_1) Y + (m_1 n_2 - m_2 n_1) Z &= 0 \\
p(n_1 - n_i) X + p(m_i - m_1) Y + (m_1 n_i - m_i n_1) Z &= 0
\end{aligned}
$$

wherefrom the angle between planes follows.

$$
\cos \varphi_i = \tag{8}
$$

$$
= \frac{p^2(n_1-n_2)(n_1-n_i) + p^2(m_2-m_1)(m_i-m_1) + (m_1 n_2 - m_2 n_1)(m_1 n_i - m_i n_1)}{\sqrt{[p^2(n_1-n_2)^2+p^2(m_2-m_1)^2+(m_1 n_2-m_2 n_1)^2][p^2(n_1-n_i)^2+p^2(m_i-m_1)^2+(m_1 n_i-m_i n_1)^2]}} \,.
$$

The analogous computation must be made for the φ'.

If the (7) for $i = 2, 3, 4, 5$ and the (8) for $i = 3, 4, 5$ are applied to the five points chosen for orientation, one can obtain the data required for the evaluation of the four unknowns Δ, Φ, Δ', Φ'. For the same points, or for all the other points whose restitution we are interested in and to which the (7) and (8) have been applied, the δ, δ' will be computed by means of (3); the latter will finally be introduced in formulae (6) which give the cartesian coordinates of the model points with reference to a tern of axes fixed to the left camera.

5. – To the analytical exposition of the procedure it is advisable to add a few remarks:

a) The method that has been followed leaves the most complete freedom in the choice of the points A_1 and A_2 which, among the orientation points, play the special role of defining the origin and the reference axis for the unknown polar coordinates of the epipoles. It must still be investigated which is the position of A_1 and A_2, so that the unknowns be determined with the best possible accuracy. It is seemed that, more than through a theoretical study, this problem can be better faced through the application of the method to experimental cases.

b) In the usual methods of analytical relative orientation the criterion used for deciding whether or not two photograms are oriented, is based on the analysis of the residual y-parallaxes on the orientation points. Here, on the contrary, the parallaxes do not appear any more, being substituted by the lack of equality of the angles in N and N′ (fig. 1), that is by the residuals of equations (1) and (2).

c) The unknowns to be determined for the relative orientation of two photograms are obviously five. However, in the present method, only four of them are computed, that is the two pairs of polar coordinates of N, N′. The fifth unknown, namely the ω rotation of the second camera around the epipolar axis NN′, resides in the condition that two homologous rays be co-planar, but does not need to be numerically computed. This fact, that may seem a little odd to the usual experience of the photogrammetrist, originates in this different approach to the problem of relative orientation which, summing up, looks for the reciprocal positions of the cameras and the epipolar axis, assuming the co-planarity of the homologous rays when computing the model coordinates.

The experience will tell whether this different choice of unknowns will be able to avoid the inconveniences that the strong correlation between the traditional parameters of the relative orientation does cause in the determination of the same and in the formation of the model.