Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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“Linear” flow-laws of elastoplasticity: a unified general approach

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **47** (1969), n.5, p. 266–276.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1969_8_47_5_266_0>

Scienza delle costruzioni. — *"Linear" flow-laws of elastoplasticity: a unified general approach.* Nota di GIULIO MAIER, presentata [(*)] dal Socio B. FINZI.

RIASSUNTO. — Si dà una formulazione generale delle leggi incrementali elastoplastiche di tipo « lineare » tra sforzi e deformazioni, ammettendo simultaneamente: non normalità (della velocità di deformazione plastica rispetto al contorno del dominio elastico), incrudimento negativo, punti singolari ed interazione tra modi plastici (cioè interferenza nelle attivazioni delle superfici di snervamento che si incontrano in un punto singolare). Si indicano e discutono delle analogie tra cui quelle coi legami incrementali forze–spostamenti, e con le leggi olonome linearizzate a tratti, ottenendo tra l'altro « modelli strutturali » per i più generali legami costitutivi formulati. Alcune salienti proprietà di questi legami, e precisamente teoremi di esistenza, di unicità e di estremo, sono enunciate in base a recenti nozioni matematiche relative al « problema di complementarietà », che, già riconosciuto come fondamentale in varie questioni di Ricerca Operativa, dalla presente ricerca risulta ricorrente e centrale anche in Plasticità.

1. FORMULATION.

The problem discussed in this paper can be more clearly formulated by referring to fig. 1. Using the traditional terminology of plasticity theory, see e.g. [1] [2], the symbols adopted can be explained as follows [(1)]. $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$

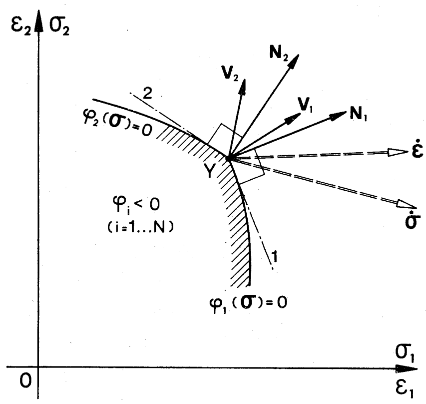

Fig. 1.

are m–vectors of *generalized* stresses and corresponding generalized strains [(2)] (their reference axes being ordinately superposed); $\dot{\boldsymbol{\sigma}}$, $\dot{\boldsymbol{\varepsilon}}$ denote their derivatives with respect to time (rates), vector OY defines the current stress state

(*) Nella seduta del 15 novembre 1969.

(1) Notation: bold face letters are used for column–vectors and matrices; a superposed tilde means transpose.

(2) When " true " stresses and (small) strains are dealt with, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{N}$ and $\mathbf{V}$, specialize to vector representations of tensors of rank two, in a 9–dimensional (or, in view of the symmetry, 6–dimensional) Euclidean space.

σ^{Y} at the yield point for the element under consideration. $\varphi_i(\sigma)$, $i = 1 \cdots N$, are *plastic functions*, such that the current elastic range in the σ–space is represented by the domain where they are all negative (3); each regular surface $\varphi_i = 0$ $(i = 1 \cdots n)$ passing through Y (" *activable* " *yielding modes*) can be replaced in an infinitesimal neighbourhood of Y with its tangent plane at Y, i.e. is defined by the outward normal $\mathbf{N}_i \equiv \left(\frac{\partial\varphi_i}{\partial\sigma}\right)_Y$ (the gradient of $\varphi_i(\sigma)$ evaluated at Y). Another vector V_i is associated at Y to each yielding mode, and gives the direction of the plastic strain vector due to the " activation " of this mode; it is assumed to be directed outward with respect to $\varphi_i = 0$ (i.e. $\tilde{\mathbf{V}}_i \mathbf{N}_i > 0$) and often interpreted as the gradient $\mathbf{V}_i \equiv \left(\frac{\partial\psi_i}{\partial\sigma}\right)_Y$ of a *plastic potential* ψ_i.

The elastic stiffness $m \times m$–matrix is denoted by $\mathbf{S}$, and assumed to be symmetric positive definite.

We shall now develop on this basis the analytical expression of a very general class of flow laws suited to govern the incremental processes which start from the given state Y. The transformation $\dot{\sigma} \to \dot{\varepsilon}$ will be called *direct* law, and $\dot{\varepsilon} \to \dot{\sigma}$ *inverse* law. We suppose that $\mathbf{S}$, $\mathbf{N}_i$ and $\mathbf{V}_i$ may be all functions of the stress and strain state and history but not of the rates. The strain rate vector is the sum of an elastic and plastic addend:

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^d \tag{1}$$

where:

$$\dot{\varepsilon}^e = \mathbf{S}^{-1}\dot{\sigma}$$

and the plastic addend is, in turn, the sum of the contributions of the n modes; hence it lies in the cone of the vectors $\mathbf{V}_i$ (see fig. 1, where $n = 2$):

$$\text{(3-a)} \quad \dot{\varepsilon}^p = \sum_1^n{}_i \mathbf{V}_i \dot{\lambda}_i \qquad \text{(3-b)} \quad \dot{\lambda}_i \geq 0 \qquad (i = 1 \cdots n)$$

where $\dot{\lambda}_i$ is a *plastic multiplier* or *activation rate* ($\dot{\lambda}_i > 0$ means that the i–th mode is activated in the sense that its relevant yielding does occur). Any yield function shall in general contain parameters depending on the plastic deformations relative to all modes: we shall take account of this fact by expressing its rate in the form:

$$\dot{\varphi}_i = \tilde{\mathbf{N}}_i \dot{\sigma} - \sum_1^n{}_j \mathrm{H}_{ij} \dot{\lambda}_j \qquad (i = 1 \cdots n) \tag{5}$$

where H_{ij} are assumed to be history but not rate dependent. H_{ij} defines how the activation of mode j influnces the yield surface $\varphi_i = 0$ of mode i in a neighbourhoord of Y: this influence results in an outward or inward

(3) The current yield locus, i.e. the boundary of this domain, is the set of points for which at least one of the N functions φ_i vanishes and none is positive.

parallel translation of the tangent plane to $\varphi_i = 0$ at Y, depending on whether $H_{ij} > 0$ or < 0 respectively. We shall call the H_{ij} *interaction coefficients* for $i \neq j$, the H_{ii} *workhardening* (or, when negative, *softening*) coefficients. If $H_{ij} = H_{ji}$ we shall speak of *reciprocity* interaction.

Since point $\sigma^Y + \dot{\sigma}\,\delta t$ cannot be outside the current elastic range:

$$\dot{\varphi}_i \leq 0 \qquad \text{for all} \quad i = 1 \cdots n . \tag{6}$$

If mode i is activated ($\dot{\lambda}_i > 0$), the stress point $\sigma(t)$ cannot leave the corresponding yield surface ($\dot{\varphi}_i = 0$); if the point leaves it ($\dot{\varphi}_i < 0$), mode i cannot be activated ($\dot{\lambda}_i = 0$) and is said to undergo unloading. Therefore

$$\dot{\varphi}_i \, \dot{\lambda}_i = 0 \qquad (i = 1 \cdots n). \tag{7}$$

Let us introduce the vectors $\tilde{\dot{\boldsymbol{\lambda}}} \equiv [\dot{\lambda}_1 \cdots \dot{\lambda}_n]$, $\tilde{\dot{\boldsymbol{\varphi}}} \equiv [\dot{\varphi}_1 \cdots \dot{\varphi}_n]$, the matrices $\mathbf{N} \equiv [\mathbf{N}_1 \cdots \mathbf{N}_n]$, $\mathbf{V} = [\mathbf{V}_1 \cdots \mathbf{V}_n]$, $\mathbf{H} \equiv [H_{ij}]$; the last one shall be referred to as *hardening interaction matrix*. On the basis of what precedes, the direct flow-law can now be given the following matrix formulation:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{S}^{-1} \dot{\boldsymbol{\sigma}} + \mathbf{V} \dot{\boldsymbol{\lambda}} (\dot{\boldsymbol{\sigma}}) \tag{8}$$

$$-\dot{\boldsymbol{\varphi}} = \mathbf{H}\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}} \quad , \quad \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \quad , \quad \dot{\boldsymbol{\varphi}} \leq \mathbf{0} \quad , \quad \tilde{\dot{\boldsymbol{\varphi}}} \dot{\boldsymbol{\lambda}} = 0 . \tag{9-a, b, c, d}$$

Eq. (8) flows from (1) (2) (3-a); (9-b, c, d) are equivalent to (3-b) (6) (7). The dependence $\dot{\boldsymbol{\lambda}}(\dot{\boldsymbol{\sigma}})$ of (8) is defined by the relation set (9), from which vector $\dot{\boldsymbol{\varphi}}$ can be clearly eliminated, and represents the crucial point.

The inverse law $\dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\varepsilon}})$ is readily obtained from the direct one by solving (8) with respect to $\dot{\boldsymbol{\sigma}}$ and substituting it in (9-a):

$$\dot{\boldsymbol{\sigma}} = \mathbf{S}\dot{\boldsymbol{\varepsilon}} - \mathbf{S}\mathbf{V}\dot{\boldsymbol{\lambda}}(\dot{\boldsymbol{\varepsilon}}) \tag{10}$$

$$-\dot{\boldsymbol{\varphi}} = (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V})\,\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\varepsilon}} \quad , \quad \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \quad , \quad \dot{\boldsymbol{\varphi}} \leq \mathbf{0} \quad , \quad \tilde{\dot{\boldsymbol{\varphi}}} \dot{\boldsymbol{\lambda}} = 0 . \tag{11-a, b, c, d}$$

The above incremental stress-strain relationship can be called "*linear*", because all rates are linearly related or constrained, except by the orthogonality requirement (α) so that the strain rates are homogeneous functions of order one of the stress rates (4). Constitutive elastoplastic laws which are not linear in the above sense, have been proposed by various authors (see [3] [4] and References in [4]).

However the present linear rate relations are most comprehensive, as they allow simultaneously for: singular points ("corners") of the yield locus, interactions between yield modes, worksoftening and deviations from normality (non association). Hence they include various extensions of the traditional theory.

(4) "Linear", if not understood in this sense, might be a misleading denomination. It is sometimes used with a narrower meaning [2]. On the other hand, mathematically speaking, both $\dot{\boldsymbol{\varepsilon}}(\dot{\boldsymbol{\sigma}})$ and $\dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\varepsilon}})$ are, clearly, not linear, as nonlinearity represents an inherent essential feature of the phenomena described.

In the rest of this paper, after a survey of the special cases covered and the relevant previous work (Sec. 2), we discuss some meaningful analogies (Sec. 3–4) and establish some basic properties of the general flow-laws in question (Sec. 5).

2. PARTICULAR CLASSES OF LAWS.

In order to characterize various special cases of increasing generality, we shall distinguish the following main features as classification criteria:

(I) Association ("normality"); $\mathbf{V} = \mathbf{N}$: (II) nonassociation: $\mathbf{V} \neq \mathbf{N}$ admitted.
(α) Regular point: $n = 1$ (hence $\mathbf{N}, \mathbf{V}$ reduce to vectors, $\dot{\lambda}$, $\dot{\varphi}$, $\mathbf{H}$ to scalars);
(β) singular point (corner) without interaction: $n > 1$, $\mathbf{H}$ diagonal;
(γ) singular point with reciprocity interaction: $n > 1$, $\mathbf{H}$ symmetric;
(δ) singular point with general interaction: $n > 1$, $\mathbf{H}$ nonsymmetric admitted.
(A) Nonsoftening: $\mathbf{H}$ nonnegative definite; (B) allowance for softening: $\mathbf{H}$ indefinite admitted (5).

The combination of hypothesis (I, α, A) characterizes the classical plastic potential theory pioneered by von Mises (1928); the less stringent property set (I, β, A) has been dealt with by Koiter [5] *et. al.* (see surveys [1] [2]): the generalization of Koiter's theory due to Mandel [6] is mainly founded on the properties (I, δ, A). In all these classes of cases a sharp distinction is traditionally made between perfectly plastic ($\mathbf{H} = \mathbf{0}$) and workhardening ($\mathbf{H} \neq \mathbf{0}$) behaviour.

Nonassociated nonsoftening flow-laws were early proposed, (Melan 1938), and in the recent years often adopted and thoroughly studied [4] both in regular and singular point, (II, α, A) and (II, β, A). Even more recently, attention was paid to softening unstable behaviour: relevant rate relations were investigated in [7] under the assumptions (II, α, B).

Experimental observations both on materials and structural elements, and the analysis of behaviour of aggregates of elements, suggest that more comprehensive descriptions would be desirable in several practical cases. These extensions, however, seem not have been investigated so far. By starting here from the weakest assumption set (II, δ, B), we aim to supply a basis for a detailed study of such extended laws.

3. ANALOGIES.

The mathematical structure of the relation set (9) can be described as follows: (9-a) maps points $\dot{\lambda}$ in the n-dimensional space R^n into points $\dot{\varphi}$ in the same space; the variables are sign-restricted by (9-b, c), so that at least

(5) The term "softening" refers to the inward motion of the tangent plane to a yield surface when this is activated, namely it means $H_{ii} < 0$. Note that the nonnegative definiteness of $\mathbf{H}$ is a sufficient but not necessary condition for ruling out softening.

one of each pair $\dot{\lambda}_i$, $\dot{\varphi}_i$ is required to vanish by the nonlinear orthogonality equation (9-d). The search for vectors $\dot{\lambda}$, $\dot{\varphi}$ solving (9) for a given vector $\tilde{\mathbf{N}}\dot{\sigma}$ is known as the (linear) "complementary problem", a problem which is at present intensively studied in operations research and mathematical programming, where it plays a key role, see e.g. [8].

We notice first that the direct (8) (9) and the inverse (10) (11) flow laws have analogous structure and the transformations $\dot{\varepsilon}(\dot{\sigma})$ and $\dot{\sigma}(\dot{\varepsilon})$ are both amenable to complementarity problems. The following further analogies are of interest for our purposes (6).

(*i*) Consider, in a known static situation, an assemblage of finite elements interconnected at certain points (nodes), where a set of additional load rates $\dot{\mathbf{F}} \equiv [F_1 \cdots F_L]$ is applied. Let the geometric II order effects be negligeable.

A case in point might be a pin-jointed truss or a finite element model of any continuous system, see e.g. [9]. Suppose that the behaviour of each element, described in terms of its "natural" generalized stresses and strains (7), conforms to the incremental laws (8) (9) (8). Let $\dot{\boldsymbol{u}}$ denote the vector of the velocities due to the load rates $\dot{\mathbf{F}}$ in their application points and directions; let $\mathbf{R}$ be the (positive definite) elastic stiffness matrix of the assemblage; $\mathbf{A}$ and $\mathbf{Z}$ are the matrices of the influence coefficients for the displacements $\boldsymbol{u}$ and, respectively, for the elastic self-stresses in the plastic elements, due to plastic strains (taken as imposed dislocations) in the same elements. By the virtual work principle, $\tilde{\mathbf{A}}$ will be the influence matrix for the elastic stresses provoked in the elements at the yield point by the loads. $\mathbf{Z}$ is symmetric, nonpositive definite [10]. Underlined symbols will indicate vectors and matrices which assemble, for all elements at the yield point, the entities which are denoted by the same symbols in (8) and (9); these entities become subvectors, or submatrices in main diagonal positions respectively. Thus the flow-laws for all elements which may undergo plastic yielding because of $\dot{\mathbf{F}}$, are simultaneously expressed, in condensed form, by the same relation set (8) and (9) with underlined symbols. Let us now formulate the problem of determining $\dot{\boldsymbol{u}}$ for given $\dot{\mathbf{F}}$.

By superposing elastic effects of loads and of inelastic strains, we may write:

$$(12) \qquad \underline{\dot{\sigma}} = \tilde{\mathbf{A}}\dot{\mathbf{F}} + \mathbf{Z}\underline{\mathbf{V}}\underline{\dot{\lambda}} \qquad\qquad (13) \qquad \dot{\boldsymbol{u}} = \mathbf{R}^{-1}\dot{\mathbf{F}} + \mathbf{A}\underline{\mathbf{V}}\underline{\dot{\lambda}}$$

(6) Analogy, as understood here, concerns the mathematical structure of relations, but does not imply that corresponding matrices possess the same properties (as rank, symmetry etc.).

(7) The specification "natural", introduced in [9], implies that stresses fulfil equilibrium, strains are unaffected by rigid body motions.

(8) If the element and its deformation patterns are homogeneous, the stress and strain space for the material is mapped into the corresponding space of the element through linear nonsingular transformations, so that the same type of law must hold for both material and element.

Substituting (12) in the relation set (9) with underlined symbols, this becomes:

$$(14\text{-a, b, c, d}) \qquad -\underline{\dot{\varphi}} = (\underline{\mathbf{H}} - \underline{\tilde{\mathbf{N}}}\mathbf{Z}\underline{\mathbf{V}})\,\dot{\underline{\lambda}} - \underline{\tilde{\mathbf{N}}}\tilde{\mathbf{A}}\dot{\mathbf{F}} \quad , \quad \dot{\underline{\lambda}} \geq \mathbf{0} \quad , \quad \underline{\dot{\varphi}} \leq \mathbf{0} \quad , \quad \tilde{\underline{\dot{\varphi}}}\dot{\underline{\lambda}} = 0 .$$

A comparison of (13) (14) which govern the transformation $\dot{\boldsymbol{u}}(\dot{\mathbf{F}})$ for the whole assemblage, with (8) (9) expressing the relation $\dot{\varepsilon}(\dot{\sigma})$ for an element, shows a complete formal analogy.

The inverse problem, i.e. the evaluation of the incremental forces required from constraints which impose on some nodes given incremental displacements, can be immediately formulated, by solving (13) with respect to $\dot{\mathbf{F}}$ and substituting in (14-a):

$$(15) \qquad \dot{\mathbf{F}} = \mathbf{R}\dot{\boldsymbol{u}} - \mathbf{R}\mathbf{A}\underline{\mathbf{V}}\,\dot{\underline{\lambda}}$$

$$(16\text{-a}) \qquad -\underline{\dot{\varphi}} = [\underline{\mathbf{H}} - \underline{\tilde{\mathbf{N}}}\,(\mathbf{Z} - \tilde{\mathbf{A}}\mathbf{R}\mathbf{A})\,\underline{\mathbf{V}}]\,\dot{\underline{\lambda}} - \underline{\tilde{\mathbf{N}}}\tilde{\mathbf{A}}\mathbf{R}\,\dot{\boldsymbol{u}}$$

$$(16\text{-b, c, d}) \qquad \dot{\underline{\lambda}} \geq \mathbf{0} \quad , \quad \underline{\dot{\varphi}} \leq \mathbf{0} \quad , \quad \tilde{\underline{\dot{\varphi}}}\dot{\underline{\lambda}} = 0 .$$

A comparison of (13) (14) with (8) (9), and of (15) (16) with (10) (11) shows a formal analogy between the relations $\dot{\boldsymbol{u}}(\dot{\mathbf{F}})$ and $\dot{\mathbf{F}}(\dot{\boldsymbol{u}})$ for the whole assemblage on one hand, and the relations $\dot{\varepsilon}(\dot{\sigma})$ and $\dot{\sigma}(\dot{\varepsilon})$ for a single element on the other.

(*ii*) Suppose that the plastic functions $\varphi_i(\sigma)$, $i = 1 \cdots N$, be linear and contain linearly, as yielding history parameters, the plastic multipliers λ_j, $j = 1 \cdots N$:

$$(17) \qquad \varphi_j = \tilde{\mathbf{N}}_i\,\sigma - \sum_{1}^{N}{}_j\, H_{ij}\,\lambda_j - K_i .$$

Thus the yield modes are represented by planes which translate by yielding. Moreover suppose that the activation of the i-th mode produces a plastic strain vector $\lambda_i \mathbf{V}_i$, where $\lambda_i \geq 0$ and $\mathbf{V}_i$ is a fixed outward vector (such that $\tilde{\mathbf{N}}_i \mathbf{V}_i > 0$). By these assumptions we obtain a *piecewise linear* constitutive law, capable of comprehending, through a suitable choice of the interaction matrix $\mathbf{H}$, various kinds of hardening rules. The complementarity requirement $\varphi_i \lambda_i = 0$ ($i = 1 \cdots N$) makes this law *holonomic*, as we pointed out in [10] though in the narrower domain of association and noninteraction. Therefore the relation set (9)

$$(18) \qquad \varepsilon = \mathbf{S}^{-1}\,\sigma + \mathbf{V}'\,\lambda$$

$$(19\text{-a, b, c, d}) \qquad -\varphi = \mathbf{H}'\,\lambda - \tilde{\mathbf{N}}'\,\sigma + \mathbf{K} \quad , \quad \lambda \geq \mathbf{0} \quad , \quad \varphi \leq \mathbf{0} \quad , \quad \tilde{\varphi}\lambda = 0$$

fully analogous to (8) (9), represents the above piecewise linear holonomic direct relation $\varepsilon(\sigma)$ between *total* (or "finite") stresses and strains. The inverse law, analogous to (10) (11), flows in an obvious way.

(9) The prime on matrices $\mathbf{V}'\ \mathbf{N}'\ \mathbf{H}'$ means that their column number now equals the number N of *all* modes.

(*iii*) Consider an aggregate of finite elements, each of which behaves according to (18) (19). The search for the overall $\boldsymbol{u}$ response to given loads $\mathbf{F}$ can be formulated following the same path which lead from (8) (9) to (13) (14). Precisely, the direct $\boldsymbol{u}\,(\mathbf{F})$ relation, in the above defined notation, reads:

$$\boldsymbol{u} = \mathbf{R}^{-1}\,\mathbf{F} + \mathbf{A}\underline{\mathbf{V}}'\,\underline{\lambda} \tag{20}$$

$$-\underline{\varphi} = (\underline{\mathbf{H}}' - \underline{\tilde{\mathbf{N}}}'\mathbf{Z}\underline{\mathbf{V}}')\,\underline{\lambda} - \underline{\tilde{\mathbf{N}}}'\tilde{\mathbf{A}}\mathbf{F} + \mathbf{K} \tag{20-a}$$

$$\underline{\lambda} \geq \mathbf{0} \quad , \quad \underline{\varphi} \leq \mathbf{0} \quad , \quad \underline{\tilde{\varphi}}\,\underline{\lambda} = 0\,. \tag{20-b, c, d}$$

(*iiii*) It seems worth mentioning that at least two other problems in different fields were previously put in the form of complementarity problems like (9) or (11) (14) (16): in dynamics, the Lagrange equations governing the motion for initial velocities of a discrete system subject to smooth unilateral constraints, ([11] sec. 5. 4., [12]); in operations research, specifically in the theory of games, the determination of " Nash equilibrium points " in bimatrix (or two person, nonzero-sum) games [13]. Despite the interest of interdisciplinary connections, we do not take here into consideration these last two analogies. On the contrary the other three might have far-reaching implications in nonlinear structural mechanics and, hence, deserve more attention.

4. Consequences of the analogies.

4.1. Postulates for general theories are often motivated, and proved to be plausible and consistent, on the ground of behaviour patterns of simple models; these, moreover, allow an easier physical insight into the theoretical developments [14] [15]. Analogy (*i*) is useful *in primis* from this standpoint, and leads straightforwardly to the remarks which follow.

1) The most general class (II, δ, B) of linear flow laws (nonassociated, softening with nonreciprocal interaction) is obeyed in the load-displacement space by a redundant assembly of elements with regular yield loci, suitable softening and nonassociation (II, α, B).

Proof. By virtue of analogy (*i*), the $\dot{\boldsymbol{u}}\,(\dot{\mathbf{F}})$ relation (13) (14) identifies with a flow-law $\dot{\varepsilon}\,(\dot{\sigma})$ (8) (2) by putting: $\mathbf{V}^* = \mathbf{A}\underline{\mathbf{V}}$, $\mathbf{N}^* = \mathbf{A}\underline{\mathbf{N}}$, $\mathbf{H}^* = \underline{\mathbf{H}} - \underline{\tilde{\mathbf{N}}}\mathbf{Z}\underline{\mathbf{V}}$. Since $\mathbf{V} \neq \mathbf{N}$, we have $\mathbf{V}^* \neq \mathbf{N}^*$ i.e. the $\dot{\boldsymbol{u}}\,(\dot{\mathbf{F}})$ law is nonassociated. The starting point in the $\mathbf{F}$–space is singular with as many modes as elements at the yield point. $\mathbf{H}^*$ is a nonsymmetric matrix, hence these modes interact without reciprocity. $\mathrm{H}^*_{hh} = \mathrm{H}_h - \tilde{\mathbf{N}}^h\mathbf{Z}^h\mathbf{V}^h$, where h is an element index: by suitable negative values of H_h, we may have $\mathrm{H}^*_{hh} < 0$, which means softening in the $\dot{\boldsymbol{u}}\,(\dot{\mathbf{F}})$ law.

Since elements of the above type in two components are easily devised (e.g. simple frictional systems), it flows from 1) that the law class (II , δ , B) is physically feasible. The statements given below rest on arguments similar to the preceding proof.

2) "Linear" flow laws of elements imply a "linear" $\dot{\boldsymbol{u}}(\dot{\mathbf{F}})$ relation for any assembly. A similar propagation property holds for the features of association and of association plus reciprocity interaction, separately.

3) In particular, an assembly of elements of the narrowest class (I, α, A) i.e. a pin-jointed truss with hardening bars, offers a model for flow-laws of class (I, β, A) (Koiter's) if it is nonredundant (i.e. $\mathbf{Z} = \mathbf{0}$), for laws of class (I, γ, A) (Mandel's) if it is redundant.

Obviously, analogy (i) is also useful in order to transfer some results from the incremental matrix analysis of structures to the study of corner flow-laws and viceversa (see an example in Sec. 5, property 5).

By virtue of the analogy (ii) the incremental elastoplastic analysis of structures with general "linear" flow laws (8) (9) and the analysis in finite terms with piecewise linear constitutive laws (18) (19) turn out to be formally and conceptually identical and to have in common extremum, existence and uniqueness theorems and methods of solution. This can be said both of the discrete matrix and the traditional continuous tensor descriptions. Holonomic piecewise linear stress-strain relations proved useful in "deformation theory" of plasticity, nonlinear elasticity and creep problems as well. In these fields analogy (iii) plays the same role as (*i*) in incremental plasticity. Thus e.g. the behaviour of pin-ended piecewise linear viscous bars can be analytically depicted in form (18) (19) with $m = 1$ (one component), $\mathbf{H}'$ diagonal with positive entries and $\mathbf{V}'=\mathbf{N}'$ reduced to a row-vector of $+1$ and -1 components, provided that ε be replaced by the time derivative $\dot{\varepsilon}$ [16]. The $\dot{\boldsymbol{u}}(\mathbf{F})$ relation for a redundant assembly of such elements is defined by (20) (21) and exhibits a symmetric positive definite (and, hence, "hardening") interaction matrix $\underline{\mathbf{H}}'^{*}$. This kind of law is thus suggested for viscous continuous materials, if they are conceived as redundant aggregates of one component elements.

5. Basic properties.

We shall state here seemingly new theorems on linear flow-laws, mainly by using some recent mathematical results on the complementarity problems, which in Sec. 3 have been seen to be central in the formulation developed in Sec. 1. The analogies allow easy extensions of our conclusions. We adopted the reasonable and weak assumption that the vectors $\mathbf{N}_1 \cdots \mathbf{N}_n$ be linearly independent.

1) There exists a unique vector $\dot{\varepsilon}$ for any given $\dot{\sigma}$ if and only if the interaction matrix $\mathbf{H}$ is a P–matrix (10).

1') There exists a unique vector $\dot{\sigma}$ for any given $\dot{\varepsilon}$ if and only if $\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V}$ is a P–matrix.

(10) A (real, square) matrix is said to be a P–matrix if all its principal minors are positive. Various characterizations of this class of matrices are given e.g. in [17].

Proof. In the direct law, as soon as problem (9) defined by $\mathbf{H}$ and $\tilde{\mathbf{N}}\dot{\sigma}$ is solved for a given $\dot{\sigma}$, vector $\dot{\varepsilon}$ uniquely flows through (8). Moreover, since $\mathbf{N}$ has full column rank, any vector of R^n can be obtained as $\tilde{\mathbf{N}}\dot{\sigma}$ from a suitable $\dot{\sigma}$ in R^m. Analogous remarks hold for the inverse law (10) (11). It was shown in [12] [18] that a complementarity problem has a unique solution for each given vector, if and only if its matrix is of class P. Statements 1) and 1′) follow.

2) The number of vectors $\dot{\varepsilon}$ is finite for all $\dot{\sigma}$, if and only if all the principal minors of $\mathbf{H}$ are nonzero.

2′) The number of vectors $\dot{\sigma}$ is finite for all $\dot{\varepsilon}$, if and only if all principal minors of $\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V}$ are nonzero.

Proof. Again the statements flow from obvious remarks on (8) (9) and (10) (11) respectively, combined with an analogous statement proved by Murty [19] for the complementarity problems.

In mechanical terms, existence and uniqueness of strain rate response to any applied stress rates mean *strict stability* (cf. [4]) or general hardening behaviour. The same facts for the stress rate response to any imposed strain rate ensure *intrisic stability* (i.e. the element can be controlled by rigid constraints) and rule out " jumps " in stresses [7]. Hence theorems 1) and 1′) yield necessary and sufficient condition for these essential mechanical features. Unbounded sets of $\dot{\varepsilon}$ responses to a given $\dot{\sigma}$ and of $\dot{\sigma}$ responses to an imposed $\dot{\varepsilon}$ characterize *perfectly* plastic behaviour and *critical softening* [7] behaviour respectively: 2) and 2′) readily supply for these occurrences necessary conditions.

3) The (any) $\dot{\varepsilon}$ corresponding to a given $\dot{\sigma}$ is defined by the $\dot{\lambda}$–solution of the following quadratic programming problem, provided that the minimum be zero:

$$(21) \qquad \min \{ \Phi = \tilde{\dot{\lambda}}\mathbf{H}\dot{\lambda} - \tilde{\dot{\lambda}}\tilde{\mathbf{N}}\dot{\sigma} \mid \dot{\lambda} \geq \mathbf{o} \ ; \ \mathbf{H}\dot{\lambda} - \tilde{\mathbf{N}}\dot{\sigma} \geq \mathbf{o} \} \ ^{(11)}.$$

An analogous statement 3′) holds for the inverse law.

Proof. An inspection of (9) shows that function Φ equals the dot product $-\tilde{\dot{\varphi}}\dot{\lambda}$, which is nonnegative on the feasible domain and vanishes only for solution of (9).

4) When $\mathbf{V} = \mathbf{N}$ and $\mathbf{H}$ is positive semidefinite, Drucker's stability postulate is complied with.

Proof. By means of (8) and (9) it is easily seen that:

$$(22) \qquad \tilde{\dot{\sigma}}\dot{\varepsilon}^p = \tilde{\dot{\lambda}}\mathbf{H}\dot{\lambda} \geq 0$$

and, by denoting with a and b two incremental processes starting from Y, that:

$$(23) \qquad (\tilde{\dot{\sigma}}_a - \tilde{\dot{\sigma}}_b)(\dot{\varepsilon}^p_a - \dot{\varepsilon}^p_b) = (\tilde{\dot{\lambda}}_a - \tilde{\dot{\lambda}}_b)\mathbf{H}(\dot{\lambda}_a - \dot{\lambda}_b) - \tilde{\dot{\varphi}}_b\dot{\lambda}_a \geq 0$$

(11) The line | means " subject to " the inequality constraints on its right, which define the " feasible domain ".

The inequalities (22) and (23) express the narrower and the extended form, respectively, of the postulate [20]. It is worth noting that the former one has its weakest (sufficient and necessary) condition in the copositiviness of $\mathbf{H}$ ($\tilde{\dot{\lambda}}\mathbf{H}\dot{\lambda} \geq 0$ for any $\dot{\lambda} \geq \mathbf{0}$). For characterizations of copositive matrices see e.g. [21].

5) When $\mathbf{V} = \mathbf{N}$, $\tilde{\mathbf{H}} = \mathbf{H}$ and $\mathbf{H}$ is positive semidefinite, the vector $\dot{\lambda}$ which defines the $\dot{\varepsilon}$ corresponding to a given $\dot{\sigma}$, is characterized by the following further extremum property:

$$\max \left\{ \Psi \equiv -\frac{1}{2}\tilde{\dot{\lambda}}\mathbf{H}\dot{\lambda} + \tilde{\dot{\lambda}}\tilde{\mathbf{N}}\dot{\sigma} \;\middle|\; \dot{\lambda} \geq \mathbf{0} \right\}. \tag{24}$$

Proof. This conclusion can be derived from [10] through the analogy *(ii)* of Sec. 3, or more directly by observing that (8) (9) can be interpreted, under the present hypotheses, as necessary and sufficient Kuhn–Tucker conditions of the concave quadratic program (24). Naturally, an analogous property holds for the inverse law.

6) When $\mathbf{V} = \mathbf{N}$, and $\tilde{\mathbf{H}} = \mathbf{H}$, if and only if $\mathbf{H}$ is positive definite, there is a unique $\dot{\varepsilon}$ for any $\dot{\sigma}$ and a unique $\dot{\sigma}$ for any $\dot{\varepsilon}$; positive semidefiniteness of $\mathbf{H}$ implies uniqueness of $\dot{\sigma}$ for any $\dot{\varepsilon}$.

Proof. In this law class, $\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N}$ is positive definite if $\mathbf{H}$ is semidefinite. A symmetric P–matrix is positive definite. These remarks and the specialization of 1) lead to 6), which can be also obtained from known theorems on the solution of quadratic programs, applied to 24) and its analogue for the inverse law.

In concluding, it is worth emphasizing that several further properties and a detailed study of the $\dot{\varepsilon}(\dot{\sigma})$ and $\dot{\sigma}(\dot{\varepsilon})$ transformations might be obtained on the present basis particularly by a more extensive use of results and solution methods concerning the complementarity problem.

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