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A characterization for the dimension of an inverse limit of compacta

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Topologia. — *A characterization for the dimension of an inverse limit of compacta.* Nota [(*)] di WŁODZIMIERZ HOLSZTYŃSKI, presentata dal Socio B. SEGRE.

SUNTO. — Se un compatto X è limite di un sistema inverso di compatti $(X_t, p_t^u)_{t \leq u \in T}$, allora sussiste la limitazione $\dim X \leq \sup (\dim X_t : t \in T)$ ed in essa non vale generalmente il segno di uguaglianza. Nel presente lavoro, usando la nozione [1] di applicazione universale, vien data una caratterizzazione della dimensione di ricoprimento di un limite inverso di compatti (teor. B).

We shall consider the covering dimension dim. Let X be a limit of an inverse system of compacta $(X_t, p_t^u)_{t \leq u \in T}$. It is a well-known fact that

$$\dim X \leq \sup (\dim X_t : t \in T)$$

and that in general the equality does not hold.

For example, let $g : H \to I$ and $h : I \to H$ be continuous mappings onto the unit closed segment and Hilbert cube respectively. Next we put

$$f_n = h \circ g \qquad \text{for} \quad n = 1, 2, \cdots.$$

Then the limit of the inverse sequence

$$H \xleftarrow{f_1} H \xleftarrow{f_2} H \xleftarrow{f_3} \cdots$$

of infinite-dimensional compacta is a one-dimensional snake-wise continuum. This is a reason for giving a characterization of dimension of an inverse limit of compacta. For such an aim, we shall use the notion of universal mapping (see [1]).

A continuous mapping $f : X \to Y$ is said to be *universal* if, for any continuous mapping $g : X \to Y$, there exists an x in X such that $g(x) = f(x)$.

Let us remark that the dimension of compact spaces is characterizated by the following theorem.

THEOREM A (see [1]). – *For a Hausdorff compact space* X, $\dim X \geq n$ *if and only if there exists a universal mapping of* X *onto the n-dimensional euclidean ball*

$$Q^n = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}.$$

For the inverse system we shall prove the following theorem.

THEOREM B. – *Let* $(X, p_t)_{t \in T}$ *be a representation of* X *as a limit of an inverse system of Hausdorff compact spaces* $(X_t, p_t^u)_{t \leq u \in T}$. *Then* $\dim X \geq n$

(*) Pervenuta all'Accademia il 25 luglio 1969.

if and only if there exists an index $s \in T$ *and a mapping* $f_s : X_s \to Q^n$ *such that* $f_s \circ p_s^t : X_t \to Q^n$ *is a universal mapping for any* $t \geq s$.

Proof. – Let $f_s : X_s \to Q^n$ be a mapping such that $f_s \circ p_s^t : X_t \to Q^n$ is universal for any $t \geq s$. Then the inverse limit $f : X \to Q^n$ of the mappings $f_t = f_s \circ p_s^t$ is, by Theorem (2.1) of [2], a universal mapping. Thus, by Theorem A, $\dim X \geq n$.

Now let us assume that $\dim X \geq n$. Then, by Theorem A, there exists a universal mapping f of X onto Q^n. We put

$$g = f \mid f^{-1}(S^{n-1}) : f^{-1}(S^{n-1}) \to S^{n-1},$$

where

$$S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in Q^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

Then there exist an index s and a mapping

$$g_s : p_s(f^{-1}(S^{n-1})) \to S^{n-1}$$

such that $g_s \circ p_s \mid f^{-1}(S^{n-1})$ is homotopically equivalent to g.

For a continuous extension $f_s : X_s \to Q^n$ the composition

$$f' = f_s \circ p_s : X \to Q^n$$

is, by Borsuk's homotopy extension theorem (see also Proposition (1.1) of [3]), a universal mapping. But

$$f' = f_s \circ p_s^t \circ p_t$$

for any $t \geq s$. Hence $f_s \circ p_s^t : X_t \to Q^n$ is a universal mapping for any $t \geq s$, and so the theorem is proved.

REFERENCES.

[1] W. HOLSZTYŃSKI, *Une generalisation du theorème de Brouwer sur les points invariants*, « Bull. Acad. Polon. Sci. », *12*, 603–606 (1964).

[2] W. HOLSZTYŃSKI, *Universal mappings and fixed point theorem*, « Bull. Acad. Polon. Sci. », *15*, 433–438 (1967).

[3] W. HOLSZTYŃSKI, *Universality of the product mappings onto products of* I^n *and snake-like spaces*, « Fund. Math. », *64*, 145–153 (1968).