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A result on section algebras over complex spaces

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Matematica. — *A result on section algebras over complex spaces.* Nota di CONSTANTIN BĂNICĂ e OCTAVIAN STĂNĂȘILĂ, presentata (*) dal Socio B. Segre.

SUNTO. — Si dà una nuova più semplice dimostrazione, tuttavia sotto condizioni leggermente più restrittive, di un risultato recente [3] relativo ai morfismi fra algebre sezioni di spazi di Stein.

Let $(X, \mathcal{O}_X)$ be a complex space having a countable base and not necessarily reduced. We note $A(X)$ the **C**–algebra of the global sections of the structural sheaf $\mathcal{O}_X$; it has a natural structure of a Fréchet algebra (when X is reduced, its topology is just the topology of the uniform convergence on every compact [4]). Let $\mathcal{O}_{X,x}$ be the fiber in the point $x \in X$ of the sheaf $\mathcal{O}_X$; we consider on it the topology of uniform convergence on neighbourhoods. Thus, the canonical map of passing to germs $A(X) \to \mathcal{O}_{X,x}$ is continuous and a sequence $f_n \in A(X)$ converges to $f \in A(X)$ if and only if the sequences $f_{n,x}$ converge to f_x in $O_{X,x}$ for any $x \in X$. Let m_x be the maximal ideal of $O_{X,x}$ and $m(x)$ the maximal ideal of $A(X)$ given by the point x.

We also recall that, if $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of complex spaces, then it induces a map of topological **C**–algebras $A(Y) \to A(X)$. We use the following two results [2]:

– The natural map $\mathrm{Hom}_{\mathbf{C}\text{-alg.}}(\mathbf{C}\{X_1, \cdots, X_n\}, A) \xrightarrow{\sim} \overbrace{m_A \times \cdots \times m_A}^{n \text{ times}}$ $\varphi \mapsto (\varphi(X_1), \cdots, \varphi(X_n))$ is bijective, for any analytic algebra A with the maximal ideal m_A (the first Hom is in the category of **C**–algebras).

– If (X, x), (Y, y) are two germs of analytic spaces, then the canonical map $\mathrm{Hom}((X, x), (Y, y)) \to \mathrm{Hom}_{\mathbf{C}\text{-alg.}}(\mathcal{O}_{Y,y}, O_{X,x})$ is bijective (the first Hom is in the category of germs and the second in the category of **C**–algebras).

For a complex space X we note $\operatorname{em}\dim X = \sup_{x \in X} \dim_{\mathbf{C}} m_x/m_x^2$.

THEOREM.—*Let* X *be a Stein space having one of the properties*:

1) *em* $\dim X < \infty$;
2) X *reduced and* $\dim X < \infty$.

Then, for any complex space Y, *any homomorphism of* **C**–*algebras* $A(X) \to A(Y)$ *is continuous.*

Proof.—According to [5] or [6], there is a closed immersion $X \to \mathbf{C}^n$ (with n suitable). Let $\mathcal{J} \subset \mathcal{O}_{\mathbf{C}^n}$ be the coherent Ideal which defines Y; then theorem B of Cartan gives the exact sequence:

$$0 \to \Gamma(\mathbf{C}^n, \mathcal{J}) \to A(\mathbf{C}^n) \to A(X) \to 0;$$

hence we reduce the problem to the case $X = \mathbf{C}^n$.

(*) Nella seduta del 15 novembre 1969.

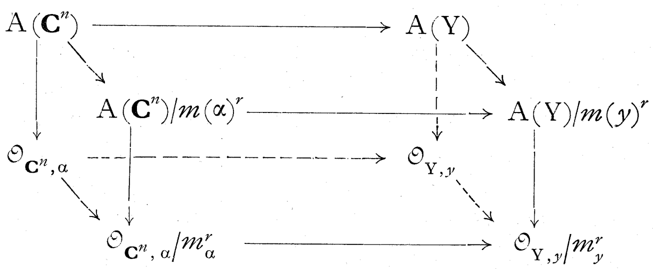
Let $\varphi : A(\mathbf{C}^n) \rightarrow A(Y)$ be a morphism of $\mathbf{C}$–algebras; for proving its continuity it suffices to show that, for any $y \in Y$, the composed map $A(\mathbf{C}^n) \rightarrow A(Y) \rightarrow \mathcal{O}_{Y,y}$ is continuous. Now, let $\alpha \in \mathbf{C}^n$ be the point of coordinates $\varphi(z_i)(y)$, where z_i are coordinate functions in $\mathbf{C}^n$ $(1 \leq i \leq n)$. For an element $f \in A(Y)$, we note $f(y)$ the image of f by the composition $A(Y) \rightarrow \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{Y,y}/m_y = \mathbf{C}$.

Obviously, $\varphi^{-1}(m(y)) = m(\alpha)$, for $m(\alpha)$ is generated by $z_i - \alpha_i$ and $\varphi(z_i - \alpha_i) \in m(y)$ $(1 \leq i \leq n)$.

Let $\varphi_y : \mathcal{O}_{\mathbf{C}^n,y} \rightarrow \mathcal{O}_{Y,y}$ be the morphism which transforms $z_i - \alpha_i$ into $\varphi(z_i)_y - \alpha_i$ for $i = 1, \cdots, n$. This morphism induces a morphism of germs $(Y, y) \rightarrow (\mathbf{C}^n, \alpha)$; hence φ_y is continuous. We have only to prove the commutativity of the following diagram:

$$\begin{array}{ccc} A(\mathbf{C}^n) & \xrightarrow{\ \varphi\ } & A(Y) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbf{C}^n,\alpha} & \longrightarrow & \mathcal{O}_{Y,y} \end{array}$$

The ring $\mathcal{O}_{Y,y}$ being noetherian, applying Krull's theorem, it is enough to show that the two morphisms are equaled by all canonical morphisms $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{Y,y}/m_y^r$ $(r \geq 1)$. Let us consider the spatial diagram:



We have to prove that the front square is commutative. But, the algebra $A(\mathbf{C}^n)/m(\alpha)^r$ being generated by all classes of polynomials in $Z_1, \cdots, Z_n$, thus the two morphisms coincide on $Z_1, \cdots, Z_n$. The theorem is thus completely proved.

Remarks. 1) In [3] this theorem is proved using the spectrum of a Stein algebra in the hypothesis $\dim X < \infty$ and Y Stein. The hypothesis relative to Y is easy to be eliminated passing to a covering for Y with Stein open sets. There are non reduced spaces X (even irreducible) of finite dimension for which em $\dim X = \infty$; hence the result from [3] is more general.

2) The theorem could be stated as follows:

The canonical map $\mathrm{Hom}_{\mathbf{C}}(A(\mathbf{C}^n), A(X)) \rightarrow \overbrace{A(X) \times \cdots \times A(X)}^{n \text{ times}}$ $\varphi \mapsto (\varphi(z_1), \cdots, \varphi(z_n))$ *is bijective.*

Indeed, from the above theorem the injectivity of this map is immediate; for the surjectivity, let $f_1, \cdots, f_n \in A(X)$. According to [2] these sections determine a morphism $X \to \mathbf{C}^n$ and the map $A(\mathbf{C}^n) \to A(X)$ induced (by this morphism) is the one looked for. Hence the behaviour of the $\mathbf{C}$–algebra $A(\mathbf{C}^n)$ relative to section algebras over complex spaces is the same as that of the polynomial algebra $\mathbf{C}[X_1, \cdots, X_n]$.

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